Example 1. *Let* $N \cong \mathbb{Z}^2 \cong M, T_N = (\mathbb{C}^*)^2, \mathscr{A} = \{m_1, m_2, m_3, m_4\} \subset M$, *with*

$$m_1 = (1,1), m_2 = (3,1), m_3 = (1,3), m_4 = (2,2).$$

M Jan. 16. 2024.

Then we have

$T_N \to \mathbb{P}^{4-1}$ 3 $\mathbb{R}^n$

$$\Psi_{\mathscr{A}} : (t_1, t_2) \mapsto (t_1t_2, t_1^3t_2, t_1t_2^3, t_1^2t_2^2) \in X_{\mathscr{A}}.$$

$\mathcal{A} = \{m_1, \dots, m_s\}$

We write down each cover:

$P = \mathrm{Conv}(\mathcal{A})$

$X_\mathcal{A}$.

$m \in \mathcal{A}$

m & P, vertex

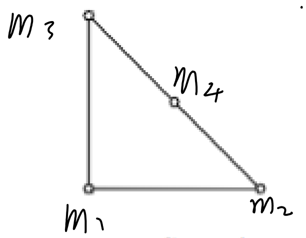


Figure 1: Set $\mathscr{A}$ in M

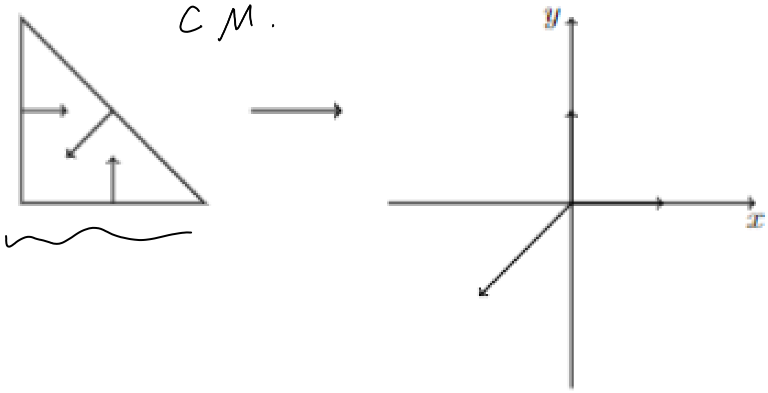
$$\begin{cases} U_1 = \textbf{\textit{Spec}}\left(\mathbb{C}[1, t_1^2, t_2^2, t_1t_2]\right) \cong \textbf{\textit{Spec}}\left(\mathbb{C}[x_2, x_3, x_4]/(x_2x_3 - x_4^2)\right), \\ U_2 = \textbf{\textit{Spec}}\left(\mathbb{C}[t_1^{-2}, 1, t_1^{-2}t_2^2, t_1^{-1}t_2]\right) \cong \textbf{\textit{Spec}}\left(\mathbb{C}[x_1, x_3, x_4]/(x_3 - x_4^2)\right), \\ U_3 = \textbf{\textit{Spec}}\left(\mathbb{C}[t_2^{-2}, t_1^2t_2^{-2}, 1, t_1t_2^{-1}]\right) \cong \textbf{\textit{Spec}}\left(\mathbb{C}[x_1, x_2, x_4]/(x_2 - x_4^2)\right), \\ U_4 = \textbf{\textit{Spec}}\left(\mathbb{C}[t_1^{-1}t_2^{-1}, t_1t_2^{-1}, t_1^{-1}t_2, 1]\right) \cong \textbf{\textit{Spec}}\left(\mathbb{C}[x_1, x_2, x_3]/(x_2x_3 - 1)\right) \end{cases}$$

After homogenization, we can get

$$X_{\mathscr{A}} = \textbf{\textit{Proj}}\left(\mathbb{C}[x_1 : x_2 : x_3 : x_4]/(x_2x_3 - x_4^2)\right) \subset \mathbb{P}^3$$

The conclusion from polytope to variety indicates that the corresponding algebraic variety for $Conv(\mathscr{A})$ *is* $\mathbb{P}^2 = \mathbb{C}[y_1 : y_2 : y_3]$.

$C\,M$. N



We have map $X_{\mathscr{A}} \to \mathbb{P}^2$ *from*

$$\mathbb{C}[y_1 : y_2 : y_3] \longrightarrow \mathbb{C}[x_1 : x_2 : x_3 : x_4]/(x_2x_3 - x_4^2)$$
$$(y_1 : y_2 : y_3) \mapsto (y_1^2 : y_2^2 : y_3^2 : y_2y_3), \quad = \dots V_2.$$

which is a variant of Veronese mapping.

Since we have the convex combination

$$m_4 = \frac{1}{2}m_2 + \frac{1}{2}m_3,$$

then

$$(m_2 - m_4) + (m_3 - m_4) = 0.$$

We can say $m_4 - m_2, m_4 - m_3 \in \mathscr{S}_4 = \mathbb{Z}_{\geq 0}\{m_1 - m_4, m_2 - m_4, m_3 - m_4\}$, *since* $m_4 - m_2, m_4 - m_3$ *can be presented by the elemnt in* $\mathscr{S}_4$. *For the points in* U_4, $\forall (x_1, x_2, x_3, x_4) \in U_4$, *we can say* $x_4 \neq 0, x_1 \in \mathbb{C}, x_2x_3 = x_4^2$, *so* $x_2 \neq 0$ *and* $x_3 \neq 0$, *that is* $(x_1, x_2, x_3, x_4) \in U_i, i = 2, 3.$

$U_4 \subseteq U_2, U_3$

$X_\mathcal{A}$. Proj

$Y_\mathcal{A}$. ~~affi~~ in Cone

From finite set $\mathscr{A}$, *we can obtain an affine cone*

$Y_\mathcal{A}$. $X_\mathcal{A}$ $\tilde{X}_\mathcal{A}$

$$Y_{\mathscr{A}} = \textbf{\textit{Spec}}\left(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1^2 - x_4, x_2x_3 - x_4^2)\right).$$

$$Y_{\mathscr{A}} = \textbf{Spec}\left(\mathbb{C}[x_1,x_2,x_3,x_4]/(x_1^2 - x_4, x_2x_3 - x_4^2)\right).$$

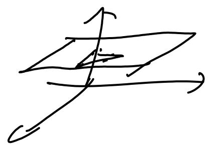
M. $\langle n, m \rangle = 1$. $\mathscr{A}$ homogen

Since the points of $\mathscr{A}$ don't distribute on a hyperplane with constant 1 and then $Y_{\mathscr{A}}$ is not induced by homogeneous ideal, we have $Y_{\mathscr{A}} \neq \widehat{X_{\mathscr{A}}}$.

Example 2. *Let $N \cong M \cong \mathbb{Z}^3, T_N = (\mathbb{C}^*)^3$, $\mathscr{A} = \{m_1, m_2, m_3, m_4\} \subset M$,*

$$m_1 = (1,1,1), m_2 = (2,1,1), m_3 = (3,1,1), m_4 = (1,2,1).$$

They're distributed on a plane $H = \{m \in M : m[2] = 1\}$. From $\mathscr{A}$, we obtain $Y_{\mathscr{A}} = \textbf{Proj}(\mathbb{C}[y_1 : y_2 : y_3 : y_4]/(y_1y_3 - y_4^2))$. From each cover:



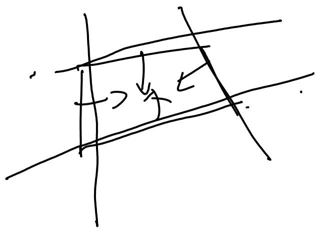
$$U_1 \cong \textbf{Spec}\left(\mathbb{C}[x_2,x_3,x_4]/(x_2^2 - x_3)\right),$$
$$U_2 \cong \textbf{Spec}\left(\mathbb{C}[x_1,x_3,x_4]/(x_1x_3 - 1)\right),$$
$$U_3 \cong \textbf{Spec}\left(\mathbb{C}[x_1,x_2,x_4]/(x_1 - x_2^2)\right),$$
$$U_4 \cong \textbf{Spec}\left(\mathbb{C}[x_1,x_2,x_3]/(x_1x_3 - x_2^2)\right).$$

So $X_{\mathscr{A}} \cong \textbf{Proj}(\mathbb{C}[x_1 : x_2 : x_3 : x_4]/(x_1x_3 - x_2^2))$, we have $Y_{\mathscr{A}} \cong \widehat{X_{\mathscr{A}}}$.

Polytope.

$S \subseteq M_{\mathbb{R}}$. $P = \text{Conv}(S)$

$\text{Dim}(P) = \text{Dim}($smallest aff subspace in $M_{\mathbb{R}}$ containing $P)$



$H_{u,b} = \{m \in M_{\mathbb{R}} : \langle m, u \rangle = b\}$ hyperplane

u: normal vector

$H^+_{u,b} = \{ \text{------} \geq b\}$ half space

Face P: $Q \leq P$, $\exists u \in N_{\mathbb{R}}$ $b \in \mathbb{R}$.

$Q = P \cap H_{u,b}$, $P \subseteq H^+_{u,b}$

$= \text{Conv}(S \cap H_{u,b})$.

Interest	Dim
facets	$\dim P - 1$
edges	1
vertices	0

Prop $P \subseteq M_{\mathbb{R}}$, polytope. Then:

(a) $P = \mathrm{Conv}(P.\ \text{vertex})$.

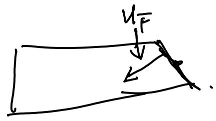
(b) If $P = \mathrm{Conv}(S)$, $P.\ \text{vertex} \subseteq S$.

(c). If $Q \leq P$, $\forall R \leq Q \Rightarrow R \leq P$.

(d). $Q < P$. $Q = \bigcap_{Q \subset F} F$. ← facets of P.

<u>Full dimension</u> $\dim(P) = \dim M_{\mathbb{R}}$.

↳ F of P, $\exists!\, H_F$, $F \subset H_F$. u_F



$$H_F = \{ m \in M_{\mathbb{R}} : \langle m, u_F \rangle = -a_F \}$$

$$H_F^+ = \{ \text{- - - - - - -} \geq -a_F \}$$

$$P = \bigcap_{F \in \text{facets}} H_F^+ = \{ m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F,\ F < P \}$$

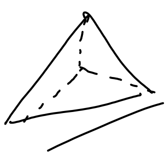
Def $P \subseteq M_{\mathbb{R}}$. $\dim(P) = d$. $d+1$ <u>points</u> vertices of P.

<u>(a). Simplex / d-simplex</u>: $P = \mathrm{Conv}(S)$. $|S| = d+1$

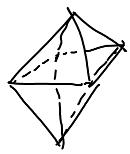
(b) Simplicial: $\forall F \in P$. facet F is simplex

(c). Simple: $\forall v \in P$, vertex, $\exists \{F_i\}_{i=1}^{d} \subset P$. facet. $v = \bigcap^{d} F_i$

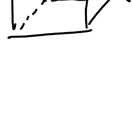
eg:



3-simplex
tetrahedron
simple, simplicial.



octahedron.
simplicial.
not simple.
not simplex



cube
not simplex
not simplicial,
simple

<u>Combinatorially equivalent</u> P_1, P_2. $P_1 \underset{\text{comb}}{\simeq} P_2$

$$\{ F_1 \leq P_1 \} \longleftrightarrow \{ F_2 \leq P_2 \}$$

preserves dim, intersection, face $\leq$.

$X_{P_1} \neq X_{P_2}$

Operation (3): $P_1 = \cdots = P_n = 0$, $(\mathbb{C}^*)^n$ roots $\approx$ Mixed Volume. Newtop

- Sum (Minkovski)

$$A_1 + A_2 = \mathrm{Conv}(S_1 + S_2)$$
$$A_1 = \mathrm{Conv}(S_1), \quad A_2 = \mathrm{Conv}(S_2)$$

$\Rightarrow k_1P + k_2P = (k_1 + k_2)P.$

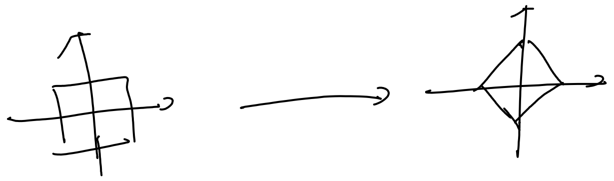
- Multiple (by $\mathbb{Z}_{\geq 0}$).

$$kP = \mathrm{Conv}(kS), \quad P = \mathrm{Conv}(S)$$

- Dual.

$$P^\circ = \{u \in N_{\mathbb{R}} : \langle m, u\rangle \geq -1,\ \forall m \in P\} \subseteq N_{\mathbb{R}}.$$

$\langle m, u_F\rangle \geq -a_F.$



Okounkov.

P° is $\mathrm{Conv}\left(\left(\frac{1}{a_F}\right) u_F : \right)$

$S \subseteq M_{\mathbb{R}}$

Lattice Polytope P $\quad S \subseteq M$, $P = \mathrm{Conv}(S)$.

eg: Standard n-simplex $\mathbb{R}^n$.

$$\Delta_n = \mathrm{Conv}(0, e_1, \cdots, e_n)$$

$$P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$$

$$\Delta_3 \underset{\text{Comb}}{\cong} P.$$

If lat.p. P full dim $\Rightarrow u_F \in N_{\mathbb{R}}$ lies in rational ray.

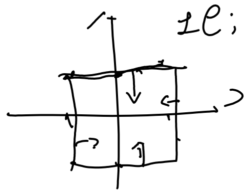
Let u_F be the ray generator, $a_F \in \mathbb{Z}$, $\langle m, u_F\rangle = -a_F$.

when $m \in F$ vertex.

$\exists$ facet presentation.

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F,\ F < P\}$$

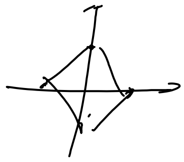
eg. $P = \mathrm{Conv}(\pm e_1, \pm e_2) \subseteq \mathbb{R}^2$.



$\pm e_i$ $\quad \langle m, \pm e_1 \rangle \geq -1 \quad$ since $a_F = 1$.

$\langle m, \pm e_2 \rangle \geq -1$.

$P^\circ = \mathrm{Conv}(\pm e_1, \pm e_2)$



Lat $P \longrightarrow P \cap M \longrightarrow X_{P\cap M}$.

$X_{P\cap M} \longrightarrow P$ × fail. when too lattice points.

eg: Δ_3 & $P = \mathrm{Conv}(0, e_1, e_2, \underline{e_1+e_2+3e_3})$

$$X_{\Delta_3 \cap \mathbb{Z}^3} = X_{P\cap\mathbb{Z}^3} = \mathbb{P}^3$$

But Δ_3 & P are different.

Def. Lat.p. $P \subseteq M_{\mathbb{R}}$ is <u>normal</u> if

$$(kP)\cap M + (lP)\cap M = ((k+l)P)\cap M.$$

$\forall k, l \in \mathbb{Z}_{\geq 0}$

$(kP)\cap M + (lP)\cap M \subseteq ((k+l)P)\cap M$ naturally.

Rmk:

· P nomal $\iff$ $\underbrace{P\cap M + \cdots + P\cap M}_{k} = (kP)\cap M$

$\dim(P) = 1 \Rightarrow P$ normal.

Def. Simplex $P \subseteq M_{\mathbb{R}}$ <u>basic</u> if P has vertex m_0 s.t. $m - m_0$, $m \neq m_0$, $\in P$ form a subset of $\mathbb{Z}$-basis of M.

eg. Δ_n is basic normal. $\qquad$ basic $\overset{?}{\Rightarrow}$ normal.

P: $\underbrace{e_1+e_2+e_3}_{\notin P+P} = \frac{1}{6}(0) + \frac{1}{3}(2e_1) + \frac{1}{3}(2e_2) + \frac{1}{3}(2e_1+2e_2+6e_3) \in 2P$.

$\Rightarrow$ P is not normal

e_1

Thm. (Normality) $P \subseteq M_{\mathbb{R}}$ full dim. lat.p. of $\dim(P) \geq 2$.
$\forall k \geq n-1$, kP is normal.

pf: Show $(kP)\cap M + P \cap M = ((k+1)P)\cap M$. $k \in \mathbb{Z}$, $k \geq n-1$

$\Rightarrow$? $((k+1)P)\cap M \subseteq (kP)\cap M + P\cap M$.

①. P simplex no interior pt. in M.

P. vertex $= \{m_1, \dots, m_n\}$.

$\downarrow$

$((k+1)P)$. vertex $= \{(k+1)m_1, \dots, (k+1)m_n\}$

$\forall m \in ((k+1)P)\cap M$.

$m = \sum \mu_i (k+1) m_i \quad \mu_i \geq 0, \ \sum \mu_i = 1$

$\mu_i (k+1) = \lambda_i$. (kP)

$m = \sum \lambda_i m_i, \quad \sum \lambda_i = (k+1)$

- $\lambda_i \geq 1, \exists i \Rightarrow m - m_i \in (kP)\cap M$, $\sum \lambda_i - 1 = k$.

  $m = (m - m_i) + m_i$ ✓

- $\forall i, \lambda_i < 1$, $n = (n-1)+1 \leq k+1 = \sum \lambda_i < n+1$.

  $\Rightarrow k = n-1$, $\sum \lambda_i = n-1$.

  $\tilde{m} = (m_1 + \dots + m_n) - m = \sum \underbrace{(1-\lambda_i)}_{\text{positive}} m_i \in P$

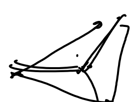
  Contradic.

② P finite union of n-dim. lat. simplex no interior.

$\mathrm{Conv}(A) = \bigcup \mathrm{Conv}(B_i)$

$B_i \subseteq A$. consisting of $\underbrace{\dim(\mathrm{conv}(A)) + 1}_{\text{"simplex"}}$, independent.

$\mathrm{Conv}(B)$ is simplex.



$\|$

$Q = \bigcup_{i=0} Q_i \qquad Q_i = \mathrm{Conv}(w_0, \dots, \hat{w_i}, \dots, w_n, v)$

D.

Cor. Every lat.p. $P \subseteq \mathbb{R}^2$ is normal.

$kP \longrightarrow (kP, k)$ $\mathbb{Z}$

Cone of P $\quad C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_{\mathbb{R}} \times \mathbb{R}$.

Lemma Let $P \subseteq M_{\mathbb{R}}$ lat.p. P is normal iff.

$(P \cap M) \times \{1\}$ generates semigroup $C(P) \cap (M \times \mathbb{Z})$

iff. $(P \cap M) \times \{1\}$ is the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$

$\mathcal{H}$

eg: $P = \mathrm{Conv}(0, e_1, e_2, e_1+e_2, e_3)$

$\mathcal{H}$ of $C(P) \cap (M \times \mathbb{Z})$ is

$(0,1)$, $(e_1, 1)$, $(e_2, 1)$, $(e_1+e_2+3e_3, 1)$, $(e_1+e_2+e_3, 2)$

$(e_1+e_2+2e_3, 2)$

Lemma. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ lat. p. $\dim \geq 2$, k_0 be the maximal height of an element of the $\mathcal{H}$. of $C(P)$. Then

(a). $k_0 \leq n-1$. (b). kP is normal for any $k \geq k_0$

Def. lat.p. $P \subseteq M_{\mathbb{R}}$ is very ample if for $\forall$ $m \in P$.vertex.

$S_{P,m} = \mathbb{Z}_{\geq 0}(P \cap M - m)$. $\longrightarrow$ $\{m'-m \mid m' \in P \cap M\}$.

is saturated in M.

Prop Normal latp P is very ample.

pf: fix $m_0 \in P$, $m \in M$. $km \in S_{P,m_0}$ $k \geq 1$.

Show $m \in S_{P,m_0}$

$$km = \sum_{m' \in P \cap M} a_{m'}(m' - m_0), \quad a_{m'} \in \mathbb{Z}_{\geq 0}.$$

$d \in \mathbb{Z}_{\geq 0}$. $\quad kd \geq \sum_{m' \in P \cap M} a_{m'}$, Then

$$km + kdm_0 = \sum_{P \cap M} a_{m'} m' + (kd - \sum_{P \cap M} a_{m'}) m_0 \in kd\,P$$

$m + d m_0 \in dP \Rightarrow m + d m_0 = \sum_1^d m_i \quad , m_i \in P \cap M . \ \forall i .$

$m = \sum (m_i - m_0) \in S_{P, m_0}$. □

Cor. Let $P \subseteq M_{\mathbb{R}}$ be the full dim lat. p. Then,

(a) If $\dim P \geq 2$. kP is very ample. $\forall k \geq n-1$

(b) If $\dim P = 2$, then. P is very ample.

Let $\underbrace{P}_{\text{lat. } M} \subseteq M_{\mathbb{R}}$, full. $\overset{=n}{\text{dim}}$. very ample p.

$P \cap M = \{m_1, \cdots, m_s\}$

$$X_{P \cap M} = \mathrm{Closure}_{\mathrm{Zar}} (\mathrm{Im} (\underbrace{T_N \to \mathbb{P}^{s-1}}))$$

$$t \mapsto (\chi^{m_1}(t), \cdots, \chi^{m_s}(t))$$

$\forall m_i \in P \cap M . \quad S_i = \mathbb{Z}_{\geq 0} (P \cap M - m_i)$

$\mathbb{P}^{s-1}$, aff open $U_i \simeq \mathbb{C}^{s-1} \quad x_i \neq 0$

$$X_{P \cap M} \cap U_i \simeq \mathrm{Spec}(\mathbb{C}[S_i])$$

$$X_{P \cap M} = \bigcup_{m_i \in \text{Vertex}} (X_{P \cap M} \cap U_i)$$

Thm. $X_{P \cap M}$ proj Toric Variety of very ample p $P \subseteq M_{\mathbb{R}}$.

P full dim $= n$. Then:

(a) $\forall m_i \in P \cap M$, affine piece $X_{P \cap M} \cap U_i$ is aff toric.

$$\underline{X}_{P \cap M} \cap U_i = U_{\sigma_i} = \mathrm{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$$

where $\underset{\dim = n}{\sigma_i}$, S. C. R. P. C. Cone $(P \cap M - m_i) \subseteq M_{\mathbb{R}}$.

(b) Torus of $X_{P \cap M}$ has char lat M. T_N.

Rmk: $\underline{\{m_i\} \subseteq M_{\mathbb{R}}}$. $\underline{M}$. $\quad X_A \cap U_i = Y_{A_i}$ affine.

pf:

$C_i = \mathrm{Cone}(P \cap M - m_i)$, $m_i \in$ Vertex.

P has supporting $H_{u,a}$. $P \subseteq H^+_{u,a}$. $P \cap H_{u,a} = \{m_i\}$

So $H_{u,0}$ is a . $0 \in C_i$, C_i is strongly convex.

$\dim C(i) = \dim(P)$. C_i, $\sigma_i = C_i^\vee$, S.C.R.R.C, $\dim = n$.

We have (?)

$S_i \subseteq C_i \cap M = \sigma_i^\vee \cap M$. Since P very ample $\Rightarrow$ $S_i \subset M$ saturated

$\boxed{\mathbb{Z}_{\geq 0}(P \cap M - m_i) \text{ saturated} \iff \mathbb{Z}_{\geq 0}(P \cap M - m_i) = \mathrm{Cone}(P \cap M - m_i) \cap M.}$

$S_i = \sigma_i^\vee \cap M$. (a) ✓.

For (b) T_N torus of U_{σ_i}, σ_i strongly convex. $\Rightarrow$

$T_N \subseteq U_{\sigma_i} = X_{P\cap M} \cap U_i \subseteq X_{P \cap M}$.

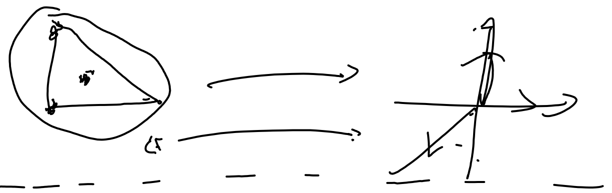
also T_N $X_{P\cap M}$ torus.

□

The normal fan.

rmk: Fan of cones $\xleftrightarrow{\text{"dual"}}$ polytopes

$0 \longrightarrow$ inding pn



P full dim. lat. p.

Faces Q. Facets F. Vertice v.

$P = \{ m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \ \forall F \}$.

$v \in P$, $\Rightarrow$ $C_v = \mathrm{Cone}(P \cap M - v) \subseteq M_{\mathbb{R}}$. $\sigma_v = C_v^\vee$.

face. $Q \subseteq P$. $v \in Q$.

$Q \longmapsto Q_v = \mathrm{Conv}(Q \cap M - v)$.

$Q_v \longmapsto Q = (Q_v + v) \cap P$.

preserve dim, inclusion intersection.

All facets C_v. $v \in P$ vertex $v \in C_v$.

$C_v = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq 0 \ F$ containing $v \}$

By duality,

$\sigma_v = \mathrm{Cone}(u_F : F$ containing $v)$

$\forall$ Faces $Q \leq P$.

$\sigma_Q = \mathrm{Cone}(u_F : F$ containing $Q)$

$\sigma_F = \mathrm{Con}(u_F)$. $\sigma_P = \{0\}$

Thm. $P \subseteq M_{\mathbb{R}}$, full dim. lat. P. and $\Sigma_P = \{ \sigma_Q \mid Q \leq P \}$

Then: (a). $\forall \sigma_Q \in \Sigma_P$ each face of σ_Q is also in Σ_P.

(b). $\sigma_Q \cap \sigma_{Q'}$, Σ_P.

$< \sigma_Q$

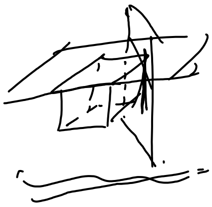
$< \sigma_{Q'}$

rmk: $\underline{\Sigma_P}$ finite collection of S.C.R.P.C. fan

normal inward-pointing. Σ_P the normal fan.

Lemma. $\underline{Q < P}$. $H_{u,b}$ supp aff hyplane of P.

Then $u \in \sigma_Q$ iff $\underline{Q} \subseteq \underline{H_{u,b} \cap P}$.



Cor. $Q \leq P$, $F < P$ facet. $u_F \in \sigma_Q$ iff $Q \subseteq F$.

Prop Q, Q', faces of full dim. lat. p $P \subseteq M_{\mathbb{R}}$.

(a). $Q \subseteq Q'$ iff $\sigma_Q \subseteq \sigma_{Q'}$.

covariant functor

~~Contravariant functor~~

Fan $\longleftrightarrow$ Polytope.

(b). $\underline{Q \subseteq Q'}$, $\sigma_{Q'}$ face σ_Q,

'dual-like'.

all faces of σ_Q are of this form

(c) $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$. Q'' smallest face of P. $Q \subseteq Q''$. $Q' \subseteq Q''$

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