2024、1.2

1. CDiv & SF.

2. Lattice & Proj Toric Varieties

3. Polytopes

4. Polytope & Toric Varieties.

5. Div & Polytope, Sheaves & Div.

Loc → Var Affine.

1. Σ. X_Σ

$\underline{(m_\sigma)}_{\sigma\in\Sigma} \longleftrightarrow \underset{D}{\mathrm{CDiv}(X_\Sigma)}$

D. $\begin{cases} m_\sigma \\ m_\sigma' \end{cases}$ $\quad M/M(\sigma)$

Prop $\mathrm{CDiv}(X_\Sigma) \cong \underline{\mathrm{Ker}\left(\bigoplus M/M(\sigma_i) \longrightarrow \bigoplus_{i<j} M/M(\sigma_i\cap\sigma_j)\right)}$

σ_i, σ_j $\quad \underline{m_{\sigma_i}|_{\sigma_i\cap\sigma_j}} \cong m_{\sigma_j}|_{\sigma_i\cap\sigma_j}$.

Support Function.

Def Σ, $N_{\mathbb{R}}$, $|\Sigma| = \bigcup \sigma \subseteq N_{\mathbb{R}}$.

(a). $\varphi : |\Sigma| \longrightarrow \mathbb{R}$. linear on each σ.

$\underline{\varphi} \in SF(\Sigma)$. $\longleftarrow$ piecewise linear function

(b). φ. integral w.r.t. lattice N.

$\underline{\varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}}$.

Denote. $SF(\Sigma, N)$.

$D = \sum a_\rho D_\rho \in CDiv_{T_N}(X_\Sigma)$, $\{m_\sigma\}_{\sigma\in\Sigma}$ CData.

$$\langle m_\sigma, u_\rho\rangle = \underset{\sim}{-a_\rho}, \quad \forall \rho \in \sigma(1)$$

RMK: $\forall f \in \Gamma(X, \mathcal{O}(D))$

$\operatorname{div}(f) + \underset{\rightsquigarrow ?}{D} \geq 0$.

Thm.

(a). D, $\{m_\sigma\}_{\sigma\in\Sigma}$, the function

$\underset{\sim}{\varphi_D :} |\Sigma| \to \mathbb{R}$

$u \mapsto \varphi_D(u) = \langle m_\sigma, u\rangle$, $u \in \sigma$.

$\underline{\text{well-defined}}$, $SF(|\Sigma|, N)$.

(b). $\varphi_D(u_\rho) = -a_\rho$, $\forall \rho \in \Sigma(1)$, so that

$$D = -\sum_\rho \varphi_D(u_\rho) D_\rho$$

(c) $D \mapsto \varphi_D$. induces an isomorphism.

$$CDiv_{T_N}(X_\Sigma) \xrightarrow{\simeq} SF(\Sigma, N).$$

pf:

Div equivalence class in $\underline{M(\sigma)}$.

$\forall \sigma$, $\varphi_D|_\sigma(u) = \langle m_\sigma, u\rangle$, $u \in \sigma$.

coincide def of CData. in SF.

(a), (b) proved.

(c). $\varphi_D \in SF(|\Sigma|, N)$, $D, E \in CDiv_{T_N}(X_\Sigma)$, $k \in \mathbb{Z}$

$\varphi_{D+E} = \varphi_D + \varphi_E$.

$\varphi_{kD} = k\varphi_D$

$$\begin{tikzcd} \underline{CDiv(X_\Sigma)} \arrow[r] & \underline{SF(|\Sigma|, N)} \end{tikzcd}$$. inj by (b).

To show surj? D: $:\varphi, \sigma$

$\underline{\varphi}$: $\mathbb{Z}_{\geq 0}$-linear function.

$\varphi|_{\sigma\cap N}: \sigma\cap N \longrightarrow \mathbb{Z}$

$\cdot\ \phi_\sigma : N_\sigma \longrightarrow \mathbb{Z}$, $N_\sigma = \mathrm{Span}(\sigma)\cap N$.

Since. $\mathrm{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z}) = M/M(\sigma)$.

$\Rightarrow \exists\, m_\sigma \in M$, $\varphi|_\sigma(u) = \langle m_\sigma, u\rangle$, $u \in \sigma$.

$D = -\sum_\rho \varphi_D(u_\rho) D_\rho \mapsto \varphi_D$. □

$$\begin{tikzcd} \underline{SF} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & Lat \\ Div \arrow[urr, leftrightarrow] \end{tikzcd}$$

$$\begin{tikzcd} & CDiv \arrow[d, phantom, "\wr"] \\ M \arrow[r] & \underline{SF(|\Sigma|, N)} \arrow[r] & Pic(X_\Sigma) \arrow[r] & 0 \end{tikzcd}$$

k-amp. ⟷ k-convex

2.

Polytope very amp ⟷ $\overset{Div}{\text{---}}$

Lat ⟶ $\underline{\text{Proj}}$ Lat ⟶ Aff

$\mathcal{A} \subseteq M$. $\mathcal{A} = \{m_1, \cdots, m_s\}$

$\Phi_{\mathcal{A}}(t) : (\chi^{m_1}(t), \cdots, \chi^{m_s}(t))$.

$k \in \mathbb{C}^*$

$\mathbb{P}^n$, $n+1$, $(x_0 : \cdots x_n) \sim (kx_0 : \cdots : kx_n)$

$T_{\mathbb{P}^n} = \{ \underbrace{(t_0 : \cdots : t_n)}_{n+1} \in \mathbb{P}^n ; t_i \in \mathbb{C}^* \}$.

$(\mathbb{C}^*)^{n+1} \xrightarrow{\pi} \mathbb{P}^n$. $\quad \underline{M_n} := \{ (a_0, \cdots, a_n) \in \mathbb{Z}^{n+1} / \sum a_i = 0 \}$

$\lambda \in \mathbb{C}^*$ $\quad (t^{a_0}, t^{a_1}, \cdots, t^{a_n}) = ((\lambda t)^{a_0}, \cdots, (\lambda t)^{a_n})$

$= \lambda^{\underset{0}{\underset{\parallel}{\boxed{\sum a_i}}}} (t^{a_0}, \cdots t^{a_n})$

$N = \mathbb{Z}^{n+1} / \mathbb{Z}(1, 1, \cdots, 1) \quad M_n^*$

$\underline{A \subseteq M}$. $\mathcal{A} = \{m_1, \cdots, m_s\}$

Def. Proj Toric Variety $X_{\mathcal{A}}$. Zariski closure in $\underline{\mathbb{P}}^{s-1}$ of the image of the map

$$T_N \xrightarrow{\Phi_{\mathcal{A}}} \mathbb{C}^s \xrightarrow{\pi} T_{\mathbb{P}^{s-1}} \subseteq \mathbb{P}^{s-1}$$

Prop. $\dim(X_{\mathcal{A}}) = \dim$ (smallest aff subspace$^{M_{\mathbb{R}}}$ containing $\mathcal{A}$)

The. affine cone of Proj Toric Variety $\underline{X_{\mathcal{A}}}$.

$\underline{Y_{\mathcal{A}}}$.

$$\begin{tikzcd} & \widehat{X_{\mathcal{A}}} \arrow[r, "\text{aff cone}"] & X_{\mathcal{A}} \\ \mathcal{A} \arrow[ur] \arrow[dr] & & \\ & Y_{\mathcal{A}} & \end{tikzcd}$$

Ideal of $Y_{\mathcal{A}}$, $\underline{0 \to L \to \mathbb{Z}^s \to M}$.
$e_i \mapsto m_i$

$I_L = \{ x^\alpha - x^\beta ; \alpha - \beta \in L \}$.

Prop. $Y_{\mathcal{A}}, \hat{X}_{\mathcal{A}}, X_{\mathcal{A}}, I_L$. TFAE;

(a) $Y_{\mathcal{A}} \subseteq \mathbb{C}^s$ is affine cone $\hat{X}_{\mathcal{A}}$ of $X_{\mathcal{A}} \subseteq \mathbb{P}^{s-1}$.

(b) $I_L = I(X_{\mathcal{A}})$

(c). I_L is homogeneous ✱

(d) $\exists u \in N, k \in \mathbb{Z}_{>0} \ni \langle m_i, u \rangle = k$. $i = 1, \dots, n$.

pf: (a) ⟺ (b), (b) ⟺ (c) obvious.

(c) ⟹ (d)

$x^\alpha - x^\beta \in I_L$, same degree

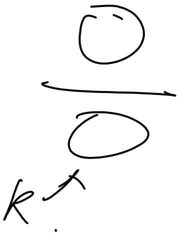
$\forall l \in L$, $l \cdot (1, \dots, 1) = 0$.
$0 : L_{\mathbb{Q}} \to \mathbb{Q}$

$\otimes \mathbb{Q}$ $\xRightarrow{\text{dual by } \mathbb{Q}}$

$$\begin{tikzcd} N_{\mathbb{Q}} \arrow[r] & \mathbb{Q}^s \arrow[r] & \mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q}) \arrow[r] & 0 \\ & (1,1,\dots,1) \arrow[u, phantom, "\in"] \arrow[ul] & & \end{tikzcd}$$

$\mathbb{Q}$

$\exists \bar{u} \in N_{\mathbb{Q}}$. $\langle m_i, \bar{u} \rangle = 1$ $k \in \mathbb{Z}_{>0}$



$k\bar{u} = u \in N$. $k \neq 0$

(d) ⟹ (a)

$Y_{\mathcal{A}}$ irreducible. $Y_{\mathcal{A}} \subseteq \hat{X}_{\mathcal{A}}$.

$$\hat{X}_A \cap \underline{(\mathbb{C}^*)^s \subseteq Y_A}.$$

Let $\underline{p}$ ✓ . $X_A \cap T_{\mathbb{P}^{s-1}}$ torus of X_A.

$$p = \mu_0(\chi^{m_1}(t), \dots, \chi^{m_s}(t)), \quad \mu_0 \in \mathbb{C}^*, \; t \in T_N.$$

$\underline{u \in N}$, $\quad \lambda^u : \mathbb{C}^* \longrightarrow T_N$.

$\qquad\qquad t \longmapsto \lambda^u(t)$

$q \in Y_A$.

$$q = (\chi^{m_1}(\lambda^u(\tau)t), \dots, \chi^{m_s}(\lambda^u(\tau)t))$$

$$= \tau^k (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$$

$k > 0$ $\quad \tau \quad p = q \in Y_A$. $\square$

A, $\longrightarrow \underline{\{m_1, \dots, m_s\}}$ $\quad (1, \dots, 1)$ $\quad \underline{M}$

$$\mathbb{Z}'A = \{\textstyle\sum a_i m_i \mid a_i \in \mathbb{Z}, \sum a_i = 0\} \underline{\subseteq M}.$$

Prop. X_A proj Toric var. of $A \subseteq M$. Then:

(a) $\mathbb{Z}'A$ is the character lattice of X_A.

(b)
$$\dim X_A = \begin{cases} \underline{\operatorname{rank} \mathbb{Z}A} - 1, & \underline{\text{if } A \text{ satisfies cond in last prop}} \; ✓ \\ \operatorname{rank} \mathbb{Z}A, & \text{otherwise.} \end{cases}$$

pf:
(a) $M \longrightarrow T_{X_{\mathcal{A}}}$, of $X_{\mathcal{A}}$

$$\begin{tikzcd}
T_N \arrow[r] \arrow[dr] & T_{\mathbb{P}^{s-1}} \arrow[r, hook] & \mathbb{P}^{s-1} \\
& T_{X_{\mathcal{A}}} \arrow[u, hook] &
\end{tikzcd}$$

$$\begin{tikzcd}
M & M_{s-1} \arrow[l] \arrow[d, two heads] \\
& M' \arrow[ul, leftrightarrow]
\end{tikzcd}$$

$\mathbb{Z}'\mathcal{A} = \mathrm{Im}(M_{s-1} \longrightarrow M)$

(b). smallest
$~~M~~$ $M \cdot \mathcal{A} \subseteq M_{\mathbb{R}}$.

$\exists\, u \in N, \quad \langle m_i, u \rangle = k, \ \forall i$

$\langle \Sigma a_i m_i , u \rangle = k\, \Sigma a_i$

$0 \to \mathbb{Z}'\mathcal{A} \longrightarrow \mathbb{Z}\mathcal{A} \xrightarrow{\langle\ ,u\rangle} k\mathbb{Z} \longrightarrow 0.$

$\underline{k > 0} \Rightarrow \underbrace{\operatorname{rank} \mathbb{Z}\mathcal{A}} - 1 = \operatorname{rank} \underline{\mathbb{Z}'\mathcal{A}} = \underline{\dim X_{\mathcal{A}}}.$

Otherwise,

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \cap M_{s-1} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \boxed{l\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{s-1} \arrow[r] \arrow[d] & \mathbb{Z}^s \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathbb{Z}'\mathcal{A}} \arrow[r] \arrow[d] & \underline{\mathbb{Z}\mathcal{A}} \arrow[r] \arrow[d] & \underline{\mathbb{Z}/l\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

$\mathrm{rk}(\mathbb{Z}'\mathcal{A}) = \mathrm{rk}(\mathbb{Z}\mathcal{A}) = \dim(X_{\mathcal{A}})$ $\square$

Affine Pieces of Proj Toric Variety

$U_i = D(X_0) = \mathbb{P}^{s-1} \setminus V(x_i)$

$\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M_{\mathbb{R}}$, $X_{\mathcal{A}}$.

$X_{\mathcal{A}} \cap U_i \qquad U_i \simeq \mathbb{C}^{s-1}$

$$(a_0, \dots, a_i, \dots a_s) \longmapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_s}{a_i}\right)$$

$$\begin{tikzcd}
a_0 \arrow[d, leftrightarrow] & a_i \arrow[d, leftrightarrow] & a_s \arrow[d, leftrightarrow] & \\
\chi^{m_0} & \chi^{m_i} & \chi^{m_s} \arrow[r] & \underbrace{\chi^{m_0-m_i}, \dots, \chi^{m_{i-1}-m_i}, \dots}
\end{tikzcd}$$

$$\boxed{\text{bias by } m_i} \longleftrightarrow \boxed{\text{affine piece of } U_i}$$

$$\begin{tikzcd}
\underline{X_{\mathcal{A}} \cap U_i} \arrow[dd] \arrow[dr, phantom, "\supseteq"] & & X_{\mathcal{A}} \cap U_j \ \forall j \arrow[dd] \\
& \underbrace{X_{\mathcal{A}} \cap U_i \cap U_j} \arrow[ur, hook] & \\
\mathrm{Spec}(\mathbb{C}[S_i]) & & \mathrm{Spec}(\mathbb{C}[S_j]) \\
\mathbb{Z}_{\geq 0}\mathcal{A} \arrow[u, "\sim"] & &
\end{tikzcd}$$

$\chi^{m_j - m_i}$ invertible.

$$X_{\mathcal{A}} \subseteq \mathbb{P}^{s-1}$$

Prop. $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$. $P = \mathrm{Conv}(\mathcal{A})$

$J = \{ j \in \{1, \dots, s\} \mid m_j$ is the vertex of $P\}$.

$$X_{\mathcal{A}} = \bigcup_{j \in J} X_{\mathcal{A}} \cap U_j$$

RMK: $\mathcal{A} \to X_{\mathcal{A}}$. convex hull.

pf: show

$$\forall i \in \underbrace{\{1, \cdots, s\}}_{\substack{\cup| \\ J}} . \quad \exists j \in J. \quad X_{\mathcal{A}} \cap U_i \subseteq X_{\mathcal{A}} \cap U_j .$$

$$P \cap M_{\mathbb{Q}} = \{ \sum_{j \in J} r_j m_j : r_j \in \mathbb{Q}_{\geq 0}, \sum r_j = 1 \}$$

For given $i \in \{1, \cdots, s\}$. $\quad m_i \in P \cap M$, $\quad m_i \xleftarrow{\text{convex combination}} m_j$

$k > 0$, $\quad k_j \geq 0$.

$$k m_i = \sum k_j m_j , \qquad \sum k_j = k .$$

$$\Rightarrow \sum k_j (m_j - m_i) = 0$$

$$\Rightarrow \text{if } k_j > 0 , \quad m_i - m_j \in S_i$$

Fix j, $\quad \chi^{m_j - m_i} \in \mathbb{C}[S_i]$ is invertible.

$$\mathbb{C}[S_i]_{\chi^{m_j - m_i}} = \mathbb{C}[S_i]$$

$$\Rightarrow X_{\mathcal{A}} \cap U_i \cap U_j = X_{\mathcal{A}} \cap U_i$$

□

50% next time.

3. Polytope

2024.1.16