2023, 12, 19

1、 How · Blow up Toric

Star subdivision. · general A.G.

1、 I $\mathcal{I}$. $\mathcal{S} = \bigoplus \mathcal{I}^d$ $\widetilde{X} = \mathrm{Proj}\, \mathcal{S} \xrightarrow{\pi} X$ noetherian.

Y

$X \setminus Y \cong \widetilde{X} \setminus Y \times \mathbb{P}^r$

$\sigma \in \Sigma$. $\pi|_{X\setminus Y}$ isomorphism

T_N - closed subset X_Σ.

$\sigma = \mathrm{Cone}(\rho_1, \cdots, \rho_s)$ $\cdot \Sigma$, $\rho_0 = \sum_{\rho \in \sigma} \rho$

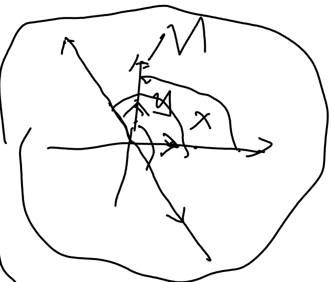
$\Sigma(\sigma) = \{\, \mathrm{Cone}(A) \mid A \subseteq \{\rho_1, \cdots, \rho_s, \rho_0\},\ \sigma \not\subseteq A \,\}$

N

$Bl_0(\mathbb{C}^2) \subseteq \mathbb{C}^2 \times \mathbb{P}^1$.

$x t_0 - y t_1$ $\quad \frac{t_0}{t_1} = \frac{y}{x} \quad \frac{t_1}{t_0} = \frac{x}{y}$

M



$\mathbb{C}[x, yx^{-1}]$

$\{x_1, \cdots, x_n\}$ $\quad y_1, \cdots, y_n$

$\mathbb{C}[xy^{-1}, y]$ $\quad x_i y_j - x_j y_i = 0.\ i, j$

$n_0 = \sum_{\rho \in \sigma} \rho$

Surface X $\quad \mathrm{Pic}(\widetilde{X}) = \mathrm{Pic}(X) \oplus \mathbb{Z}$

$\mathrm{Pic}(X)$

$$\begin{cases} \cdot\ C \cdot D = \pi^*(C) \cdot \pi^*(D) \\ \cdot\ C \cdot E = 0. \\ \cdot\ E^2 = -1 \end{cases}$$

$\square$

2、 $\mathrm{Pic}(X \times X) \ncong \mathrm{Pic}(X) \times \mathrm{Pic}(X)$

X = Elliptic

$0 \to p_1^* \mathrm{Pic}(X) \oplus p_2^* \mathrm{Pic}(X) \to \mathrm{Pic}(X \times X) \to \mathrm{End}(X, P_0) \to 0$

$\ni f$

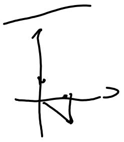
$f : X \to X \quad f(P_0) = P_0$

In Toric $\cong$

eg (2).

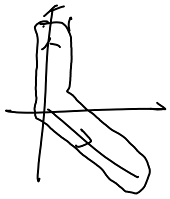
1. C_d. $\sigma = \mathrm{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$.

$d=3$  $Cl(U_\sigma) \cong \mathbb{Z}/d\mathbb{Z}$ $\quad \rho_i \to D_i$

Σ_1 

$[D_1] = [D_2]$. since $Pic(U_\sigma) = 0$

$\Rightarrow [D_1], [D_2]$ is not Cartier if $d > 1$.

Σ_2  $X_{\Sigma_2} \cong C_d \setminus \{\gamma_\sigma\}$ smooth

$Pic(X_{\Sigma_2}) \cong Cl(X_{\Sigma_2})$ □

2. $X = V(xy - wz) \subseteq \mathbb{C}^4$, $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subseteq \mathbb{R}^3$

$u_i \to e_i \quad i = 1,2, \quad u_3 = e_1+e_3. \quad u_4 = e_2+e_3.$

$u_1 + u_4 = u_2 + u_3 \qquad D_i \sim u_i.$

$a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4 \in CDiv \Leftrightarrow a_1 + a_4 = a_2 + a_3$

$Cl(X) = \mathbb{Z}$, since $Pic(X) = 0$. $\quad D_i \notin CDiv$

□

Smooth, Simplicial → orbifold.

Div ~ Smooth.

1. $\Sigma \subseteq N_{\mathbb{R}}$ fan.

(a). Σ smooth if $\sigma \in \Sigma$ is smooth,

$\mathrm{Cone}(e_1, \cdots, e_r)$ $+ \cdots + e_n$ N.

$\mathbb{Z}$-basis

(b) Σ simplicial if $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$. $\forall \sigma \in \Sigma$, $\sum_{\rho \in \sigma(1)} \mathbb{R}\rho = N_{\mathbb{R}}$

cone σ $e_1, \cdots, e_r$. $\mathbb{R}$. $N_{\mathbb{R}}$.

linear independent

Thm. smooth $\longrightarrow$ Alg geo (a) X_Σ is smooth iff Σ is smooth.

Convex geo ? U_σ

$\sigma \longleftrightarrow$

(b) X_Σ is an orbifold iff Σ is simplicial.

pf: (a). X_Σ smooth $\iff$ $\forall \sigma \in \Sigma$. U_σ smooth.

Lemma. $\sigma \subseteq N_{\mathbb{R}}$. S.C.R.P.C maximal cone, $\leftarrow$ dual.
$T_{p_\sigma}(U_\sigma)$ — Zariski Tangent space. $(m/m^2)^*$
to U_σ at p_σ. $\leftarrow$ fixed point.

$$\dim T_{p_\sigma}(U_\sigma) = |\mathcal{H}| \xrightarrow{\mathbb{Z}\text{-gen}} \sigma^\vee \cap M.$$

pf: $m \in \mathbb{C}[S_\sigma]$. $m = \{\chi^m \mid m \in S_\sigma \setminus \{0\}\}$.

χ^m. $m = \left(\bigoplus_{m \in \mathcal{H}} \chi^m\right) \oplus m^2$. $\sim U_\sigma$. $(\mathcal{O}_{p_\sigma, U_\sigma}, m_\sigma)$

$$\boxed{\dim m/m^2 = |\mathcal{H}|}.$$

$$\boxed{m/m^2 \xrightarrow{\cong} m_\sigma/m_\sigma^2}$$

$\square$

$\dim T_{p_\sigma}(U_\sigma)$ for any affine embedding $U_\sigma \to \mathbb{C}^L$
lower bound of L.

Thm. σ S.C.R.P.C. in N. Then U_σ smooth $\iff$ σ smooth.

(U_σ smooth $\iff$ σ smooth)

pf: If smooth, $\Rightarrow$ normal

$\Leftarrow$ U_σ $\sigma = \text{Cone}(e_1, \dots, e_r)$. $\sigma^\vee = \text{Cone}(e_1, \dots, e_r, \pm e_{r+1}, \dots, \pm e_n)$

$\hookrightarrow \mathbb{C}[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_n^{\pm 1}]$

$\leftrightarrow \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$

$\Rightarrow$ $\underset{\text{smooth}}{U_\sigma} \xrightarrow{} \text{smooth}$, $\dim(\sigma) = n = \dim N_{\mathbb{R}}$. σ smooth $\Leftarrow U_\sigma$.

$p_\sigma \in U_\sigma$, fixed point.

$\dim T_{p_\sigma}(U_\sigma) = |\mathcal{H}| \leq$

σ has n edges, $\mathcal{H}$ generate $\sigma^\vee$

$M = \mathbb{Z} S_\sigma$, $\sigma^\vee$, $\sigma^\vee$ generators $\xrightarrow{\text{basis}} \mathbb{Z}^n \cong M$,

$\Rightarrow \sigma^\vee$ is smooth, $(\sigma^\vee)^\vee$ is smooth.

$\dim(\sigma) = r < n$. $\sigma \subseteq N_1 \subset N$, N/N_1 is torsion free.

N_2.

$N = N_1 \oplus N_2 \longleftrightarrow M = M_1 \oplus M_2$.

$$\underset{\substack{\downarrow \\ U_{\sigma, N_1}}}{\sigma \subseteq (N_1)_{\mathbb{R}}} \Rightarrow \underset{\substack{\downarrow \\ U_{\sigma, N}}}{\sigma \subseteq N_{\mathbb{R}}} \hookrightarrow S_{\sigma, N_1} \subseteq M_1, \ S_{\sigma, N} \subseteq M.$$

$\dim$: r, n.

$\Downarrow$

$S_{\sigma, N} = S_{\sigma, N_1} \oplus M_2$.

$\mathbb{C}[S_{\sigma, N}] = \mathbb{C}[S_{\sigma, N_1}] \oplus \mathbb{C}[M_2]$

$\updownarrow$

$\underset{\text{smooth}}{U_{\sigma, N_1}} \times T_{N_2} = U_{\sigma, N}$. $\forall p$. p_1 p_2

$\square$

$\square$

Prop. X_Σ, Σ. $N_{\mathbb{R}} \cong \mathbb{R}^n$, $\sigma \in \Sigma$ $\dim(\sigma) = n$, then $\mathrm{Pic}(X_\Sigma)$ free abelian group.

pf: $\forall D \in \mathrm{CDiv}_{T_N}(X_\Sigma)$, kD, $k > 0$. $\exists \chi^m$.

$\mathrm{div}(\chi^m) = kD$

Suppose $D = \sum_\rho a_\rho D_\rho$.

Let $\dim(\sigma) = n$. Since $D \in CDiv(X_\Sigma) \Rightarrow D \in CDiv(U_\sigma)$
$\sim U_\sigma$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$ affine

Orbit-Cone Correspondence, we

$$D|_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho.$$

$\exists m' \in M$. $\qquad \underset{\operatorname{div}(\chi^{m'})|_{U_\sigma}}{\parallel} \Rightarrow a_\rho = \langle m', u_\rho \rangle, \ \forall \rho \in \sigma(1)$

$$\Rightarrow k a_\rho = \langle m, u_\rho \rangle \quad , \forall \rho \in \sigma(1)$$

$$\langle k m', u_\rho \rangle = k a_\rho = \langle m, u_\rho \rangle \ , \forall \rho \in \sigma(1)$$

Since $\{u_\rho\}$ span $N_{\mathbb{R}}$. since $\dim(\sigma) = n \Rightarrow km' = m$.;
$D = \operatorname{div}(\chi^{m'})$ $\square$

Prop. Let X_Σ, Σ TFAE:

(a). $Div(X_\Sigma) = CDiv(X_\Sigma)$

(b). $Cl(X_\Sigma) = Pic(X_\Sigma)$

(c) X_Σ is smooth.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & Div \arrow[r] & Cl \arrow[r] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[u] & CDiv \arrow[r] \arrow[u] & Pic \arrow[r] \arrow[u] & 0 \\ & N \arrow[u] & & & \end{tikzcd}$$

pf: $(a) \Leftrightarrow (b)$ / (a) or $(b) \Rightarrow (c)$ Span

$\because$ (a) or (b) Every $Div(X_\Sigma) \subseteq CDiv(X_\Sigma)$

Let $U_\sigma \subseteq X_\Sigma$. open affine. Since $(U_\sigma \subseteq X_\Sigma)$

$$Cl(X_\Sigma) \twoheadrightarrow Cl(U_\sigma)$$

? if every Weil divisor on U_σ is Cartier.

Using $Pic(U_\sigma) = 0$, & exact sequence.

$$M \longrightarrow Div_{T_N}(U_\sigma) = \bigoplus_{\rho \in \sigma(1)} \mathbb{Z} D_\rho \qquad \sigma. \ N.$$

$\sigma(1) = \{\rho_1, \dots, \rho_s\}$.

$$M \longrightarrow \mathbb{Z}^s$$
$$m \longmapsto (\langle m, u_{\rho_1} \rangle, \dots, \langle m, u_{\rho_s} \rangle)$$

$\Phi : \mathbb{Z}^s \longrightarrow N$ by. $\Phi(a_1, \cdots, a_s) = \sum a_i u_{\rho_i}$

$\Phi^* : M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}^s, \mathbb{Z}) = \mathbb{Z}^s$

Need to show.

Φ^* surjective $\Leftrightarrow$ Φ injective & $N/\Phi(\mathbb{Z}^s)$ is torsion free

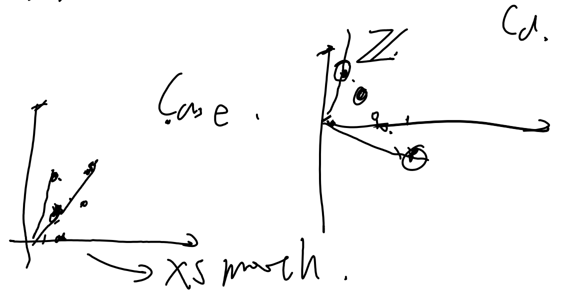
$\Leftrightarrow$ $u_{\rho_1}, \cdots, u_{\rho_s}$ can be extended basis of N

smooth of σ

① $\Leftarrow$.

Φ^* is surjective

Case. Cd. $\mathbb{R}$, not



$\mathbb{Z}$

Prop X_Σ., TAEF

(a). Positive integer multiple that is CDiv.

(b). $\mathrm{Pic}(X_\Sigma)$ has finite index in $Cl(X_\Sigma)$

(c). X_Σ is simplicial

pf: above.

Description of $\mathrm{CDiv}(X_\Sigma)$

Thm. Σ, X_Σ, $D = \sum_\rho a_\rho D_\rho$, TFAE:

(a) $D \in \mathrm{CDiv}_{T_N}(X_\Sigma)$

(b) D is principal on U_σ, $\forall \sigma \in \Sigma$.

(c). $\forall \sigma \in \Sigma$, $\exists m_\sigma \in M$, $\langle m_\sigma, u_\rho \rangle = -a_\rho$. $\forall \rho \in \sigma(1)$

(d) $\forall \sigma_{max} \in \Sigma$, $\exists m_\sigma \in M$, $\boxed{\langle m_\sigma, \cdots}$ ——

$\Sigma(n)$

Furthermore if $D \in \mathrm{CDiv}_{T_N}(X_\Sigma)$ is as in part (c)

(1) m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$.

(2). $\tau < \sigma$, $m_\sigma \equiv m_\tau \mod M(\tau)$.

pf:

$D|_{U_\sigma} \Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) ($\Leftrightarrow$ (d))

(d) $\Rightarrow$ (c) $\forall \tau \in \Sigma$, $\exists \sigma \in \Sigma_{max}$, $\tau < \sigma$, if $m_\sigma \in M$, works for σ. also works for all faces of σ.

(1). $m_\sigma \in M$, $\langle m, u_\rho \rangle = -a_\rho$ $\forall \rho \in \sigma(1)$

Then we have.

$\langle m'_\sigma, u_\rho \rangle = -a_\rho$, $\forall \rho \in \sigma(1)$

$\Leftrightarrow$ $\langle m'_\sigma - m_\sigma, u_\rho \rangle = 0$, $\forall \rho \in \sigma(1)$

$\Leftrightarrow$ $\langle m'_\sigma - m_\sigma, u \rangle = 0$, $u \in \sigma$.

$\Leftrightarrow$ $m'_\sigma - m_\sigma \in \sigma^\perp \cap M = M(\sigma)$

$\Downarrow$

unique in mod $M(\sigma)$

$m_\sigma \equiv m_\tau \mod M(\tau)$.

$\square$

Polytope ~ proj ~

2 week.

Jan. 2.