2023.12.5

Outline:

1. Two remaining questions from last time.
   1.1 Minimality of σ, in Orbit-Cone Correspondence.
   1.2. Ideal of $V(\tau) = \overline{O(\tau)}$ in U_σ.
2. Weil divisor
   2.1 Valuation function
   2.2 Rational function & div
   2.3. Exact sequence of Div.
   2.4. Some example (5);
3. Cartier divisor
   3.1 Exact sequence of CDiv.
   3.2. Affine case
   3.3. Some example (2).
   3.4. $Pic(X_\Sigma)$ abelian case.
   3.5. Smooth ~ Pic/Cl; Simpliciality ~ Pic/Cl.
   3.6. Description of CDiv.

1. 1.1 Orbit-Cone Thm.

Minimal $\sigma \in \Sigma$. ① finite ② T_N-inv orbit.

U_σ is T_N-inv affine cover.

$O \subseteq U_\sigma$.

1.2.

① $I \longleftrightarrow V(\tau) \subseteq U_\sigma$, with $V(\tau) = \overline{O(\tau)}$, $p \in V(\tau)$.

$T_N \cdot \gamma_\tau$ $\quad t \in T_N$. $\quad t \cdot \gamma_\tau \in V(\tau)$

$I = \langle \chi^m : m \in (\tau^\perp) \cap (\sigma)^\vee \cap M \rangle \subseteq \mathbb{C}[S_\sigma]$

$f \in$

$(\tau^\perp)^c$

$\gamma_\tau = m \mapsto \begin{cases} 1 & m \in \tau^\perp \cap S_\sigma \\ 0 & \text{otherwise} \end{cases}$

$f(p) = 0$

② $V(\tau)$ closure of Zariski set.

by Chevally Thm. $O(\tau)$ is constructible

Zariski = Euclidean

2. Weil divisor

k-dim cone $\sigma \in \Sigma$.

$\dim(O(\sigma)) = n - k$. by Orbit–Cone.

$\Sigma(1)$ 1-dim cones (rays) of Σ.

$\forall \rho \in \Sigma(1) \Rightarrow$ codim 1 orbit $O(\rho)$

$V(\rho) = \overline{O(\rho)}$ is T_N-inv prime divisor on X_Σ.

$D_\rho := V(\rho)$. Weil $\xrightarrow{\text{abel}}$ prime. { codim 1, closed, integral

$\longrightarrow$ valuation.

$\nu_\rho := \nu_{D_\rho} : \mathbb{C}(X_\Sigma)^* \longrightarrow \mathbb{Z}$

$\forall \rho \in \Sigma(1)$. a minimal generator $u_\rho \in \rho \cap N$.

$\forall m \in M$, $\chi^m : T_N \longrightarrow \mathbb{C}^*$. $\in \mathbb{C}(X_\Sigma)^*$

generic pt ξ $f \in \mathcal{O}_{X,\xi}$ T_N.

① Let X_Σ be the toric variety of fan Σ. If $\rho \in \Sigma(1)$ has minimal generator u_ρ and χ^m is character $\longleftrightarrow$ $m \in M$. then

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle$$

pf: $u_\rho \in N$, extend a basis of N

$e_1 = u_\rho, e_2, \cdots, e_n \in N$.

$N = \mathbb{Z}^n$, $\rho = \mathrm{Cone}(e_1) \subseteq \mathbb{R}^n$

$U_\rho = \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]) = \mathbb{C} \times (\mathbb{C}^*)^{n-1}$

and $D_\rho \cap U_\rho$ is $V(x_1) := \{x_1 = 0\}$.

DVR: $$\mathcal{O}_{X_\Sigma, D_\rho} = \mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]_{\langle x_1 \rangle}$$

$\forall f \in \mathbb{C}(x_1, \cdots, x_n)^*$. $\quad V_\rho(f) = l \in \mathbb{Z}$.

$$f = x_1^l \frac{g}{h}, \quad g, h \in \mathbb{C}[x_1, \cdots, x_n] \setminus \langle x_1 \rangle$$

Relate this to $\nu_\rho(\chi^m)$, we get the dual basis

$$e_1 = u_\rho, \ e_2, \ \cdots \ e_n \in N.$$

$\forall m \in M$.

$$\chi^m = x_1^{\langle m, e_1 \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle}$$

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle. \qquad \square.$$

② Rational function. $\underline{\mathrm{div}}$. $\mathcal{O} \to \mathrm{Div}$

Prop: $\forall m \in M$, χ^m is a rational function on X_Σ, and its divisor is given by

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

pf: $\quad$ ← Orbit–Cone Correspondence

$$D_\rho \longleftrightarrow X \setminus T_N.$$

Since χ^m is defined and nonzero of T_N.

It follows that $\mathrm{div}(\chi^m)$ is supported on

$$\bigcup_{\rho \in \Sigma(1)} D_\rho.$$

$$\Rightarrow \mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} V_{D_\rho}(\chi^m) \cdot D_\rho$$

$$= \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle \cdot D_\rho. \quad \text{by last result} \qquad \square$$

Div of toric

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \quad T_N\text{-inv on } X_\Sigma.$$

:

Thus.

$$\mathrm{Div}_{T_N}(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \subseteq \mathrm{Div}(X_\Sigma)$$

③. Exact Sequence

Prop: $\underline{M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma) \longrightarrow 0.}$

$$m \longmapsto \mathrm{div}(\chi^m)$$

$$D \longmapsto [D]$$

Furthermore, we have

$$0 \to M \to \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma) \longrightarrow 0.$$

iff $\{u_\rho : \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

pf: (since $\bigcup_{i=1}^{s} D_i = X \setminus U$, ∴

$$\bigoplus \mathbb{Z} D_i \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0)$$

$\downarrow$

$$\mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma) \longrightarrow Cl(T_N) \longrightarrow 0$$

Since $\mathbb{C}[x_1, \dots, x_n]$ is UFD $\Rightarrow$ $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ UFD

$\cong \mathbb{C}[M] \leftrightarrow T_N$.

$\Rightarrow$ $Cl(T_N) = 0$.

$$\underline{M} \to \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \underline{Cl(X_\Sigma)}.$$

is zero. since. $\underbrace{m \mapsto \chi^m}$ $\Rightarrow$ $\left\{ \begin{array}{l} \mathrm{im}(M \to \mathrm{Div}_{T_N}(X_\Sigma)) \\ \subseteq \ker(\mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma)) \end{array} \right.$

Now suppose that $D \in \mathrm{Div}_{T_N}(X_\Sigma)$. $D \to [0] \in Cl(X_\Sigma)$

$\Rightarrow D = \mathrm{div}(f)|_{T_N} = 0$

Regard as an element of $\mathbb{C}(T_N)^*$.

f has 0 divisor on T_N, so that $f \in \mathbb{C}[M]^*$.

$\Rightarrow f = c\chi^m$ $c \in \mathbb{C}^*$ and $m \in M$.

(undetermined coefficient ? $f = \sum a_i \chi^{m_i}$)

D. move T_N — Boundary

It follows that X_Σ..

$$D = \mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m)$$

Const $\xrightarrow{\times}$ pole or zero

Finally, $m \in M$, $\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$

is the zero divisor.

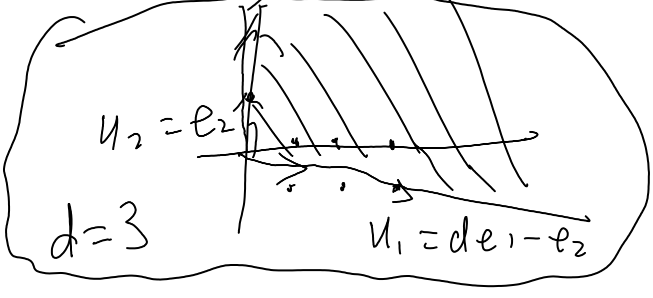
$\Rightarrow \forall \rho \in \Sigma(1)$, $\langle m, u_\rho \rangle = 0$.

$\Rightarrow m = 0$, when u_ρ span $N_\mathbb{R}$.

Conversly. $\boxed{\text{exact}}$ u_ρ must span $N_\mathbb{R}$. □

③ Example (5)

1/5. $\sigma = \mathrm{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$.



U_σ is rational normal cone.

$$\overset{M}{\mathbb{Z}^2} \longrightarrow \overset{\mathrm{Div}_{T_N}(U_\sigma)}{\mathbb{Z}^2}, \quad A = \begin{bmatrix} d & -1 \\ 0 & 1 \end{bmatrix}$$

$\underline{Cl(\hat{C}_d) \cong \mathbb{Z}/d\mathbb{Z}}$

$0 \sim \operatorname{div}(\chi^{e_1}) = \langle e_1, u_1 \rangle D_1 + \langle e_2, u_2 \rangle D_2$

$= d D_1$

$0 \sim \operatorname{div}(\chi^{e_2}) = \langle e_2, u_1 \rangle D_1 + \langle e_2, u_2 \rangle D_2$

$= -D_1 + D_2$

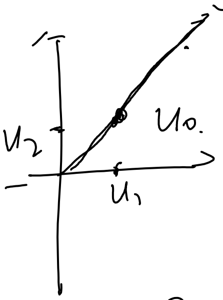
$M = \langle e_1, e_2 \rangle$.

$Cl(C_d) = [\operatorname{div} \chi^m] \quad m \in M$

$= \langle [D_i] : d[D_i] = 0 \rangle$

$\cong \mathbb{Z}/d\mathbb{Z}$.

2/3. <u>blow-up of $\mathbb{C}^2$ at O</u>.



σ. $u_1 = e_1$, $u_0 = e_1 + e_2$. $u_2 = e_2$

$\in \mathbb{C}^2$

$\gamma_\sigma := (0,0)$

D_1, D_2, D_0

* <u>Exceptional!</u>

$0 \sim \operatorname{div}(\chi^{e_1}) = D_1 + D_0$

$0 \sim \operatorname{div}(\chi^{e_2}) = D_2 + D_0$

$\Rightarrow Cl(Bl_0(\mathbb{C}^2)) \cong \mathbb{Z}$

$[D_1] \sim [D_2] \sim [-D_0]$.

3/5 $\mathbb{P}^n$, $u_0 = -e_1 - e_2 - \cdots - e_n$

$u_i = e_i$

$M \to \operatorname{Div}_{T_N}(\mathbb{P}^n)$

$\mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1}$

$(a_1, \cdots, a_n) \longmapsto (-a_1 - \cdots - a_n, a_1, \cdots, a_n)$

We get $\mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}$

$(b_0, \cdots, b_n) \longmapsto b_0 + \cdots + b_n$.

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$Cl(\mathbb{P}^n) \cong \mathbb{Z}$.

$\mathbb{P}^2$, hyper plane.

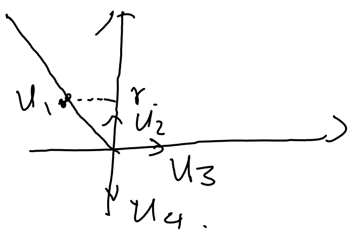
4/5, $X_{\Sigma_1} \times X_{\Sigma_2}$

$$Cl(\mathbb{P}^m \times \mathbb{P}^n) \cong Cl(X_{\Sigma_1} \times X_{\Sigma_2})$$
$$\cong Cl(X_{\Sigma_1}) \oplus Cl(X_{\Sigma_2}).$$

not True

- Elliptic Curve.
  Ell × Ell.

5/5 $\mathcal{H}_r$: $u_1 = -e_1 + re_2$, $u_2 = e_2$, $u_3 = e_1$, $u_4 = -e_2$



$D_i \longleftrightarrow u_i$

$0 \sim div(\chi^{e_1}) = -D_1 + D_3$

$0 \sim div(\chi^{e_2}) = rD_1 + D_2 - D_4$.

$Cl(\mathcal{H}_r) \cong \mathbb{Z}^2$. □

3. Cartier Divisor.

A CDiv is a Weil Div.

$\Rightarrow$ $D \sim \sum a_\rho D_\rho$, $a_\rho \in \mathbb{Z}$

$\Rightarrow \in$ CDiv.

Denote. $CDiv_{T_N}(X_\Sigma) \subseteq Div_{T_N}(X_\Sigma)$

Since. $\forall m \in M$, $div(\chi^m) \in CDiv_{T_N}(X_\Sigma)$

3.1. Prop. $M \longrightarrow CDiv_{T_N}(X_\Sigma) \longrightarrow Pic(X_\Sigma) \longrightarrow 0$.
$m \longmapsto div(\chi^m)$

$$D \longmapsto [D]$$

Furthermore, we have a short exact sequence.

$$0 \to M \to \mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$$

iff $\{u_\rho : \rho \in \Sigma(1)\}$ span $N_{\mathbb{R}}$.

Rmk: In general case: X

$\mathrm{Pic}(X) = \mathrm{Cl}(X)$ iff X is noetherian, geometric integral, separated, locally factorial.

- $\mathbb{C}[U_\sigma]$ is UFD $\Rightarrow$ $\mathbb{C}[U_\sigma]_{(p)}$ is UFD (Kaplanski Thm)
- U_σ is separated.
- $\mathbb{C}[U_\sigma]$ integral.
- $\mathbb{C}[M]$ is Noetherian $\Rightarrow$ $\mathbb{C}[S]$ as well.

$$\mathrm{Pic}(U_\sigma) \cong \mathrm{Cl}(U_\sigma)$$

3.1. Prop. $\sigma \subseteq N_{\mathbb{R}}$ S.C.P.C ($\mathbb{R}$?)

(a). $\forall D \in \mathrm{CDiv}_{T_N}(U_\sigma)$ is the $\mathrm{div}(\chi^m)$, $m \in M$.

(b). $\mathrm{Pic}(U_\sigma) = 0$.

Rmk: ($X = \mathrm{Spec}(\mathbb{Z}[\sqrt{-5}])$)

pf:

Let $R = \mathbb{C}[\sigma^\vee \cap M]$, effective $D = \sum_\rho a_\rho D_\rho \in \mathrm{CDiv}_{T_N}(U_\sigma)$, $a_\rho \geq 0$.

$\Rightarrow$ $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}(-D))$

$:= \{f \in K : f = 0$ or, $f \neq 0$, $\mathrm{div}(f) \geq D\} = I$.

ideal in R.

I is T_N-inv,

$$I = \bigoplus_{\chi^m \in I} \mathbb{C}\cdot\chi^m = \bigoplus_{\mathrm{div}(\chi^m)\geq D} \mathbb{C}\cdot\chi^m.$$

Orbit-Cone Correspondence: $\rho \in \sigma(1)$

$$\Rightarrow \underbrace{O(\sigma) \subseteq \overline{O(\rho)} = D_\rho}.$$

Thus $O(\sigma) \subseteq \bigcap_\rho D_\rho$

Now fix $p \in O(\sigma)$. Since CDiv is locally principal, $\exists\, \mathcal{U}, p\in\mathcal{U}$, principal in $\mathcal{U}$.

$$\mathcal{U} = (U_\sigma)_h = \mathrm{Spec}(R_h),\ h\in R,\ h(p)\neq 0.$$

Thus

$$\exists f \in \mathbb{C}(U_\sigma)^*,\ \underbrace{D|_{\mathcal{U}} = \mathrm{div}(f)|_{\mathcal{U}}}.$$

D is effective. $f\in R_h$. h is invertible on $\mathcal{U}$

we may assume $f\in R$. Then,

$$\mathrm{div}(f) = \underbrace{\sum v_{D_\rho}(f) D_\rho} + \underbrace{\sum_{E\neq D_\rho} v_E(f) E}.$$

$$\geq \sum_\rho v_{D_\rho}(f) D_\rho = D.$$

We can write $f = \sum a_i \chi^{m_i}$, $a_i\in\mathbb{C}^*$, $\mathrm{div}(\chi^{m_i}) \geq D$.

Restricting to $\mathcal{U}$.

$$\Rightarrow \mathrm{div}(\chi^{m_i})|_{\mathcal{U}} \geq \mathrm{div}(f)|_{\mathcal{U}}.$$

$$\Rightarrow \chi^{m_i}/f \text{ is morphism on } \mathcal{U}.$$

$$1 = \frac{\sum a_i \chi^{m_i}}{f} = \sum a_i \frac{\chi^{m_i}}{f}.$$

$\underset{\text{previously}}{p \in U}$. $\frac{\chi^{m_i}}{f}(p) \neq 0 \Rightarrow p \in V \subseteq U$. $\frac{\chi^{m_i}}{f}(V)$ nonvanish

$\Rightarrow \operatorname{div}(\chi^{m_i})|_V = \operatorname{div}(f)|_V = D|_V$.

Since $\operatorname{div}(\chi^{m_i})$, D have support contained.

$\bigcup_\rho D_\rho$. $D_\rho \cap V \neq \emptyset$, $\forall \rho$ $(p \in V)$

$\Rightarrow$ We have $\operatorname{div}(\chi^{m_i}) = D$.

For (a). $D \in \operatorname{CDiv}_{T_N}(U_\sigma)$. since $\dim(\sigma^\vee) = \dim M_{\mathbb{R}}$

we can find. $m \in \sigma^\vee \cap M$. s.t.

$\langle m, u_\rho \rangle > 0$, $\forall \rho \in \sigma(1)$

$\Rightarrow$ $\operatorname{div}(\chi^m)$ is positive linear combination of D_ρ.

$\Rightarrow$ $D' = D + \operatorname{div}(\chi^{km}) \geq 0$. , k large enough $\in \mathbb{Z}_{\geq 0}$,

$\Rightarrow$ D' is the divisor of a character.

So it same for D. $\square$.

(b) ✓

2. eg.

2023.12.19. 6: pm..