2023.11.21

Outline:

1. Two results on affine case.
   - Points in affine toric varieties
   - Fixed points on U_σ.
   - Eg

2. T_N-orbit & Cone-Orbit correspondence.
   - Eg
   - Distinguished Pt.
   - T_N-orbit $O(\sigma)$
   - Correspondence Thm.

3. T_N-Weil divisor.
   - Valuation
   - Rational Function
   - $M \to \mathrm{Div}_{T_N}(X_\Sigma) \to Cl(X_\Sigma) \to 0$
   - Eg

4. T_N-Cartier divisor (If we have time)

1.

1.1 Points in affine toric variety

Prop $V = \mathrm{Spec}(\mathbb{C}[S])$, S affine semigroup

(a.) $pt \in V$. (b) $m \in \mathbb{C}[S]$. (c) $\underset{\text{semi-group}}{\mathrm{Hom}}(S, \mathbb{C})$.

pf:

(a) $\to$ (c).

$\forall p \in V$, we have

$$\underbrace{\begin{aligned} S &\longrightarrow \mathbb{C} \\ m &\longmapsto \chi^m(p) \end{aligned}}_{\text{semigroup homomorphism}}, \quad \chi^m \in \mathbb{C}[S] = \mathbb{C}[V]$$

(c) $\to$ (b) $\leftrightarrow$ (a)

$\gamma : S \to \mathbb{C}$, $\forall \chi^m \in \mathbb{C}[S]$, we let

$\bar{\gamma}(\chi^m) = \gamma(m)$

$\bar{\gamma} : \mathbb{C}[S] \to \mathbb{C}$ $\quad \mathbb{C}$-Alg

$\mathrm{Ker}(\bar{\gamma})$ is maximal idea in $\mathbb{C}[S]$

$A = \{m_1, \cdots, m_s\}$ $\quad S = \mathbb{Z}_{\geq 0} A$, $V = V_A$.

Let $p = (\gamma(m_1), \cdots, \gamma(m_s)) \in \mathbb{C}^s$.

$(\chi^\alpha - \chi^\beta)(p) = 0.$, $\alpha = (a_1, \cdots, a_s)$
$\beta = (b_1, \cdots, b_s)$

$$\sum a_i m_i = \sum b_i m_i$$

$$\prod \gamma(m_i)^{a_i} = \gamma(\sum a_i m_i) = \gamma(\sum b_i m_i)$$
$$= \prod \gamma(m_i)^{b_i}$$

$\Rightarrow p \in V.$ $\square$

1.2. Torus action

$T_N \curvearrowright V$: $\quad t \in T_N$, $p \in V$, $\gamma : S \to \mathbb{C}$.

$\Rightarrow t \cdot p := (m \mapsto \chi^m(t)\gamma(m))$

Prop V $=\mathrm{Spec}(\mathbb{C}[S])$ affine toric variety

(a) T_N action has a fixed point iff S is strong convex, (i.e. $S \cap (-S) = \{0\}$).

$$S \longrightarrow \mathbb{C}$$
$$m \longmapsto \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases}$$

(b) $V = Y_A \subseteq \mathbb{C}^s$, $A \subseteq S \setminus \{0\} \Rightarrow$

$\exists$ T_N - fixed point $\Leftrightarrow$ $0 \in Y_A$. the unique fixed pt is 0.

pf: (a) $\forall p \in V \longleftrightarrow \gamma_p : S \to \mathbb{C}$ $\quad \gamma_{t \cdot p} = t \cdot \gamma_p$

$t \cdot p$. $\quad T_N \cdot \{p\} = \{p\} \Leftrightarrow \chi^m(t)\, \gamma_p(m) = \gamma_p(m)$

$$\gamma_p = \begin{cases} 1 & m=0 \\ 0 & m\neq 0. \end{cases}$$

γ_p is semigroup homomorphism $\Leftrightarrow$ S strongly convex

(b). $V = Y_A \subseteq \mathbb{C}^s$ has fixed pt $S = \mathbb{Z}_{\geq 0} A$.

$$p \longleftrightarrow \gamma_p = \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases}$$

$\Rightarrow p = 0 \in \mathbb{C}^s \Rightarrow 0 \in Y_A$

Converse

$0 \in \mathbb{C}^s$, $(\mathbb{C}^*)^s \cdot \{0\} = \{0\}$

$\Rightarrow \underbrace{(Y_A \cap (\mathbb{C}^*)^s)}_{T_N \text{ of } Y_A} \cdot \{0\} = \{0\}$ $\square$

Cor σ S.C.R.P.C in $N_{\mathbb{R}}$, $\underline{U_\sigma}$ has T_N-fixed

iff $\underline{\dim\sigma = \dim N_{\mathbb{R}}}$

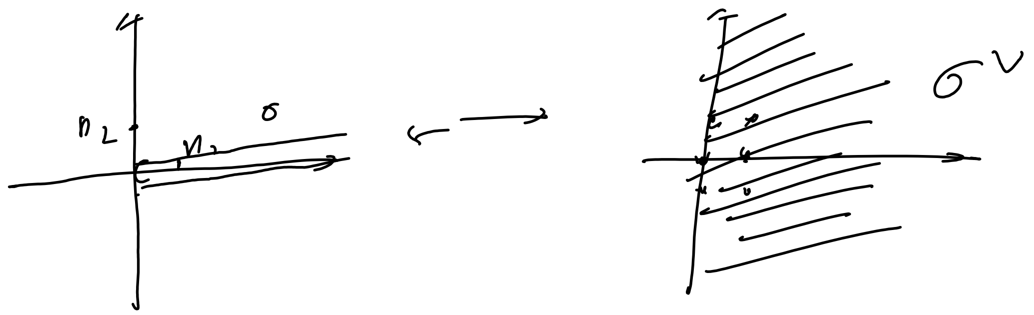
$\langle \chi^m \mid m \in S_\sigma \setminus \{0\} \rangle = \mathbb{C}[S_\sigma]$

where $S_\sigma = \sigma^\vee \cap M$. □

eg. $N = \mathbb{Z}^2$, $\langle n_1, n_2 \rangle$.

M $\langle m_1, m_1 \rangle$

$\sigma = \mathbb{R}_{\geq 0} n_1$, $\sigma^\vee = \mathbb{R}_{\geq 0} m_1 + \mathbb{R} m_2$



$S_\sigma = \sigma^\vee \cap M = \mathbb{Z}_{\geq 0} m_1 + \underline{\mathbb{Z}_{\geq 0} m_2 + \mathbb{Z}_{\geq 0}(-m_2)}$

$\exists\, \gamma_p \longleftrightarrow p$ fixed pt.

$\gamma_p(0) = 1$, $\gamma(m_2) = 0$.

but $1 = \gamma_p(0) = \gamma_p(m_2 + (-m_2))$

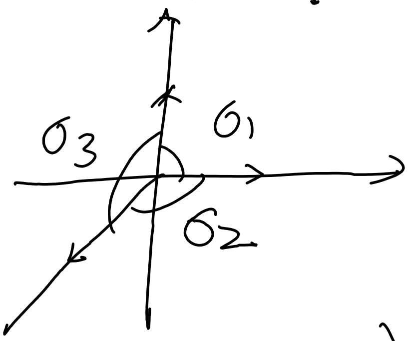
$= \gamma_p(m_2)\,\gamma_p(-m_2) = 0$. □

2.

2.1

$\underline{e}$. $\boxed{X}$

eg. $\mathbb{P}^2 \leftarrow \sigma_\cdot \to \sigma^\vee \hookrightarrow \sigma^\vee \cap M$



Σ, $T_N = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$

$u = (a,b) \in \mathbb{Z}^2$,

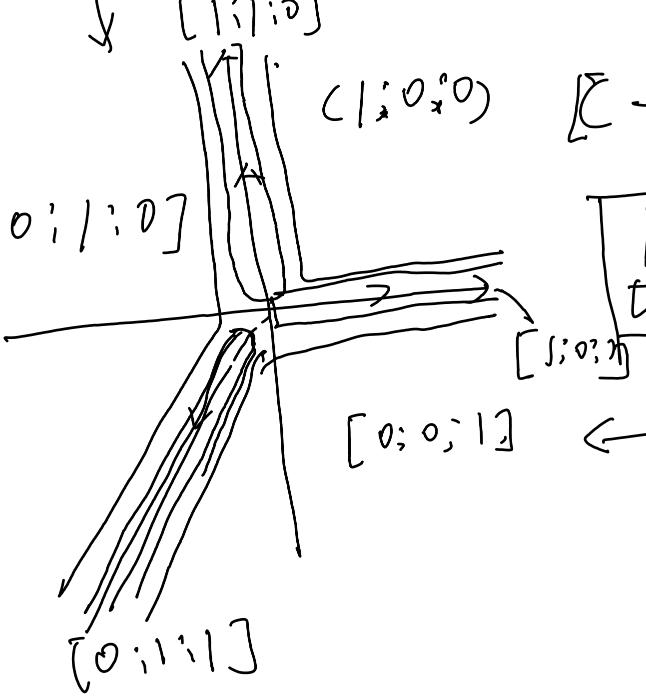
$\lambda^u(t) = \underbrace{[1;t^a;t^b]}_{} \in \mathbb{P}^2$

$\wr$

$[t;t^{a+1};t^{b+1}]$

$(\ ,\ ,\ ) \in \mathbb{A}^3$

$[\ ;\ ;\ ]$



$\boxed{\lim_{t\to 0} \lambda^u(t)}$ $u \in \mathbb{Z}^2$.

$[0;0;1] \not\longleftrightarrow$ $(-1,1)$

$[1, t^{-1}, t]$

$\wr$

$[t, 1, t^2]$

2.2.

Prop Let $\sigma \subseteq N_{\mathbb{R}}$, S.C.R.P.C, $u \in N$

$u \in \sigma \iff \exists \lim_{t\to 0} \lambda^u(t) \in U_\sigma$.

Moreover, if $u \in \mathrm{RelInt}(\sigma)$, then.

$$\underset{\sim}{\gamma_\sigma} = \lim_{t\to 0} \lambda^u(t)$$

Def Distinguished Point of U_σ

$m \in S_\sigma$. $\gamma_\sigma = \begin{cases} 1, & m \in S_\sigma \cap \sigma^\perp = \sigma^\perp \cap M. \\ 0, & \text{otherwise} \end{cases}$

pf: $\lim\limits_{t\to 0} \lambda^u(t) \in U_\sigma \Longleftrightarrow \lim\limits_{t\to 0} \chi^m(\lambda^u(t)) \in \mathbb{C},\ \forall m \in S_\sigma$

$\Longleftrightarrow \lim\limits_{t\to 0} t^{\langle m, u\rangle} \in \mathbb{C},\ \forall m \in S_\sigma = \sigma^\vee \cap M$

$\Longleftrightarrow \langle m, u\rangle \geq 0,\ \forall m \in S_\sigma$

$\Longleftrightarrow u \in (\sigma^\vee)^\vee = \sigma$

$$\lim_{t\to 0} \lambda^u(t) \longleftrightarrow \begin{array}{rcl} S_\sigma & \longrightarrow & \mathbb{C} \\ m & \longmapsto & \lim\limits_{t\to 0} t^{\langle m, u\rangle} \end{array}$$

$\| $

γ_σ.

$u \in \mathrm{RelInt}(\sigma) \Rightarrow \langle m, u\rangle > 0,\ m \in S_\sigma \setminus \sigma^\perp$

$\langle m, n\rangle = 0,\ m \in S_\sigma \cap \sigma^\perp$ $\square$

eg: $V = V(xy - zw) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ γ_σ.

$\sigma^\vee = \mathrm{Cone}(m_1, m_2, m_3, m_1 + m_2 - m_3)$

$(t_1, t_2, t_3) \longmapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \in V$

$N = \mathbb{Z}^3_{\geq 0} \ni u = (a, b, c)$

$\lambda^u(t) = (t^a, t^b, t^c, t^{a+b-c})$

$u \in \mathrm{RelInt}(\sigma) \Rightarrow a, b, c > 0,\ a+b-c > 0$

$\lim\limits_{t\to 0} \lambda^u(t) = (0,0,0,0) = \gamma_\sigma$. $\square$

2,3 T_N-orbit of σ.

$$\mathcal{O}(\sigma) = T_N \cdot \gamma_\sigma \subseteq X_{\Sigma} \quad \Sigma \to \text{fan}$$

Lemma. σ, S.C.R.P.C in $N_{\mathbb{R}}$.

$N_\sigma := \langle \sigma \cap N \rangle$, $N(\sigma) := N/N_\sigma$.

(a) perfect pairing:

$$\langle\, ,\, \rangle : \sigma^\perp \cap M \times \underset{n}{N(\sigma)} \longrightarrow \mathbb{Z}$$

induced by the dual pairing $\langle\, ,\, \rangle : M \times N \to \mathbb{Z}$.

(b) The pairing of part (a) induces a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}$$

where $T_{N(\sigma)} := N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$, is torus associated to $N(\sigma)$.

pf:

(a) m, or n.

$n \mapsto \langle m, n \rangle$. $\quad m \mapsto \langle m, n \rangle$ isomorphism.

$\underbrace{\langle m, n \rangle}_{n}$ $\quad$ ∎

(b) by (a)

Lemma. Let σ be a S.C.R.P.C in $N_{\mathbb{R}}$.

$$\mathcal{O}(\sigma) = \{\gamma : S_\sigma \to \mathbb{C} ; \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$$
$$\cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}$$

pf:

The set

$$\mathcal{O} := \{\gamma : S_\sigma \to \mathbb{C} : \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\}$$

contains γ_0 and is invariant under the action of T_N

Show $\mathcal{O} \cong \mathrm{Hom}_\mathbb{Z}$.

$\sigma^\perp$ is the largest vector space of $M_\mathbb{R}$ contained in $\sigma^\vee$.

$\sigma^\perp \cap M$ is a subgroup of S_σ.

If $\gamma \in \mathcal{O}$, $\gamma|_{\sigma^\perp \cap M}$ yields

$\hat{\gamma} = \sigma^\perp \cap M \longrightarrow \mathbb{C}^*$ Group Hom

$\gamma \in \mathcal{O}$.

$$\gamma(m) = \begin{cases} \hat{\gamma}(m), & m \in \sigma^\perp \cap M. \\ 0, & \text{otherwise}. \end{cases}$$

It follows $\mathcal{O} \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0$$

$\otimes_\mathbb{Z} \mathbb{C}^*$

$\Downarrow$

$$T_N = N \otimes_\mathbb{Z} \mathbb{C}^* \twoheadrightarrow T_{N(\sigma)} = N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^*$$

kernel

Bijections

$$T_{N(\sigma)} \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*) \cong O$$

are compatible with the T_N-action.

$T_N \cdot O$ transitively on O.

$$\gamma_\sigma \in O \Rightarrow O = T_N \cdot \gamma_\sigma = O(\sigma)$$ □

2.4.

Thm (Orbit-Cone Correspondence)

X_Σ be the toric variety, Σ fan in $N_{\mathbb{R}}$. Then:

(a) $\{\text{cones in } \Sigma\} \xleftrightarrow{1-1} \{T_N\text{-orbit in } X_\Sigma\}$

$$\sigma \longleftrightarrow O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}^*)$$

(b) Let $n = \dim N_{\mathbb{R}}$. $\forall \sigma \in \Sigma$,

$$\dim(O(\sigma)) = n - \dim(\sigma)$$

(c)

$$U_\sigma = \bigcup_{\tau < \sigma} O(\tau)$$

(d). $\tau < \sigma$ iff $O(\sigma) \subseteq \overline{O(\tau)}$ $\begin{cases} \text{classical} \\ \text{Zariski} \end{cases}$

$$\overline{O(\tau)} = \bigcup_{\tau < \sigma} O(\sigma)$$

pf: O is T_N-orbit in X_Σ, $X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$.

$U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$.

$\sigma \in \Sigma$ is minimal, $\exists!\ O \longleftrightarrow O(\sigma)$

$y \in O$, $m \in S_\sigma$, s.t. $y(m) \neq 0 \Rightarrow m$ in face of $\sigma^\vee$.

But the face of $\sigma^\vee$, have form

$\sigma^\vee \cap \tau^\perp$, $\tau < \sigma$.

$\begin{cases} 1.\ \text{face of } \sigma^\vee \cap M \\ 2.\ \tau^\perp \cap \sigma^\vee \end{cases}$

$\exists\, \tau < \sigma$, s.t.

$$\underbrace{S = \{m \in S_\sigma : y(m) \neq 0\}}_{O(\sigma)} = \sigma^\vee \cap \tau^\perp \cap M.$$

$\Rightarrow$ $y \in U_\tau$, $\tau = \sigma$ (Minimality of σ)

Prop Appendix of Oda.

Let C be a C.P.C in V, A subset $S \subseteq C$ is a face of C iff S satisfies the following conditions

(a) S is a convex cone.

(b) $v + v' \notin S$, $\forall\, v \in C \setminus S$, $v' \in C$.

($C \setminus S$ is an "ideal" in the additive semigroup C)

(a) Convex ✓.

(b) $y \in S$, $y' \in C \setminus S$. $y \cdot y' = 0$.

$S = \sigma^\perp \cap M$,

$O = O(\sigma)$ $\begin{cases} \text{equal} \\ \text{disjoint.} \end{cases}$

(b). $\dim(O(\sigma)) = \dim(\mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*))$

$= \dim(T_N(\sigma))$

exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underbrace{N_\sigma} & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0 \\
\text{dim} \downarrow \text{rank} & & \downarrow & & \downarrow & & \downarrow & & \\
& & \dim(\sigma). & & \dim(N)=n. & & \dim(T_{N(\sigma)}) & &
\end{array}$$

$\dim O(\sigma) = n - \dim(\sigma)$. □

(c). $\underline{U_\sigma = U \text{ orbits}}$.

If $\tau < \sigma$, $O(\tau) \subseteq U_\tau \subseteq U_\sigma$.

$\Rightarrow$ $O(\tau) \subseteq U_\sigma$.

(d). $\begin{cases} \text{classical} \\ \text{Zariski} \end{cases}$

Classical $\overline{O(\tau)}$ T_N-invariant. , union of orbits.

1. $O(\tau)$ T_N-invariant.
2. $\{p_n\} \subseteq O(\tau)$, $p = \lim p_n \in \overline{O(\tau)}$

$T_N \cdot p = \lim T_N \cdot p_n \in \overline{O(\tau)}$

$O(\sigma) \in \overline{O(\tau)} \Longrightarrow O(\tau) \subseteq U_\sigma$.

Otherwise, $O(\tau) \cap U_\sigma \underset{(a)}{=} \emptyset \Rightarrow \overline{O(\tau)} \cap U_\sigma = \emptyset$

U_σ is open in classical Topology.

Once $O(\tau) \subseteq U_\sigma \Rightarrow \tau < \sigma$.

Assume $\tau < \sigma$. To prove $O(\sigma) \subseteq \overline{O(\tau)}$

To show $\overline{O(\tau)} \cap O(\sigma) \neq \emptyset$

γ_τ semigroup homomorphism of distinguished point of U_τ.

$$\gamma_\tau(m) = \begin{cases} 1, & m \in \tau^* \cap M, \\ 0, & \text{otherwise.} \end{cases}$$

Let $u \in \mathrm{RelInt}(\sigma)$, $\forall t \in \mathbb{C}^*$,

$$\gamma(t) = \lambda^u(t) \cdot \gamma_\tau.$$

Homomorphism $\downarrow$

$$m \mapsto \chi^m(\lambda^u(t))\, \gamma_\tau(m) = t^{\langle m, u\rangle} \gamma_\tau(m)$$

$\gamma(t) \in O(\tau)$, $\forall t \in \mathbb{C}^*$. $O(\tau) \overset{\text{by def}}{=} T_N \cdot \gamma_\tau$.

Let $t \to 0$. since $u \in \mathrm{RelInt}(\sigma)$

$$\langle m, u\rangle = \begin{cases} > 0 & m \in \sigma^\vee \setminus \sigma^\perp \\ 0 & m \in \sigma^\perp. \end{cases}$$

$$\gamma(0) := \lim_{t \to 0} \gamma(t) \in U_\sigma, \quad \exists\, pt \in O(\sigma).$$

It's in the closure of $O(\tau)$, so that

$$O(\sigma) \cap \overline{O(\tau)} \neq \emptyset$$

Zariski $\checkmark$.

U_τ. $\quad U_\tau \cap \overline{O(\tau)} \longleftrightarrow V(I) \subset \mathbb{C}[S_\tau]$.