correct

- 2 error last time. 2023. 11. 7.
- Some example about $T_N\operatorname{emb}(\Delta)$ ← Fan.
- General case about toric variety (David. Cox)
- T_N - orbit

- 1. <u>Hausdorff</u> about $T_\Delta\operatorname{emb}(\Delta)$

$\vdash$ $\pi: U_\tau \longrightarrow U_{\sigma_1} \times U_{\sigma_2}$. $\tau = \sigma_1 \cap \sigma_2$. closed.

$\pi^*: \mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] \longrightarrow \mathbb{C}[S_\tau]$

$e(m) \otimes e(n) \longmapsto e(m+n)$

$(\chi^m \otimes \chi^n \longmapsto \chi^{m+n})$ ✓.

$\boxed{S_\tau = S_{\sigma_1} + S_{\sigma_2}}$ π^* is surjective.

$\mathbb{C}[S_\tau] \cong (\mathbb{C}[S_{\sigma_1}] \otimes \mathbb{C}[S_{\sigma_2}])/\ker\pi^*$

$\operatorname{Im}\pi \subseteq U_{\sigma_1} \times U_{\sigma_2}$ □

- 2.

$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$

We have. $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_0)$

$(\Leftarrow)$.

$\forall m \in S_\tau$, $\exists a \in \mathbb{Z}_{\geq 0}$, $\Rightarrow m + a m_0 \in \sigma^\vee$.

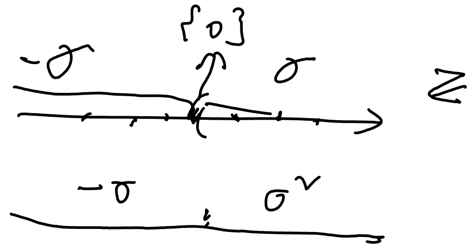
$\Rightarrow$ $\hookrightarrow \in M \cap \sigma^\vee = S_\sigma$.

$\Rightarrow S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$. □

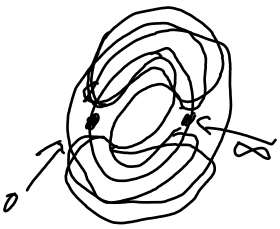
eg:

1. $N=\mathbb{Z}$, $\sigma=\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$.

$\Delta := \{\sigma, -\sigma, \{0\}\}$



$T_N\,\mathrm{emb}(\Delta)=\mathbb{P}^1$.



$\mathbb{C}[x]$

$U_\sigma \cong \mathbb{C}$, $U_{-\sigma} \cong \mathbb{C} \leftarrow \mathbb{C}[y]$

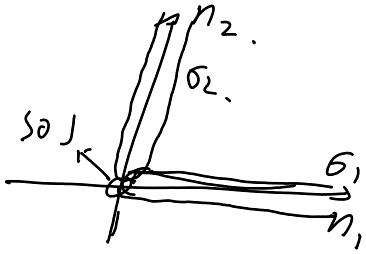
$U_{\{0\}} = T_N = \mathbb{C}^*$

$y=\frac{1}{x}$ $\quad \mathbb{C}[x,y]/(xy-1)$

2. $\{n_1, n_2\}$ $\mathbb{Z}$-basis $N\cong\mathbb{Z}^2$

$\{m_1, m_2\}$ ------ M.

$\Delta := \{\sigma_1, \sigma_2, \{0\}\}$



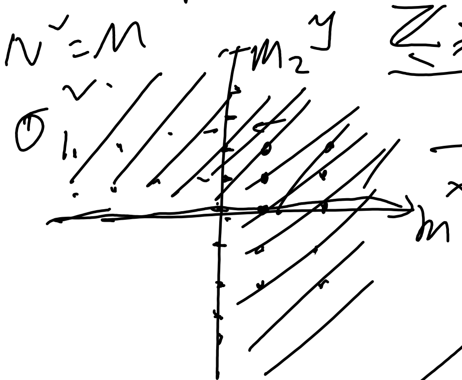
$\sigma_1 := \mathbb{R}_{\geq 0} n_1$

$\sigma_2 := \mathbb{R}_{\geq 0} n_2$

$\sigma_1^\vee := \mathbb{R} m_2$ $\quad \sigma_2^\vee := \mathbb{R} m_1$

$\cap M$ $\quad +\mathbb{Z}_{\geq 0} m_1$ $\quad\quad +\mathbb{Z}_{\geq 0} m_2$

$\mathbb{Z}_{\geq 0} m_2 + \mathbb{Z}_{\geq 0}(-m_2)$ $\quad \mathbb{Z}_{\geq 0} m_1 + \mathbb{Z}_{\geq 0}(-m_1)$



$\to \sigma_2^\vee$ $\quad \cap M$ $\to \mathbb{C}[x, x^{-1}, y]$

$U_{\sigma_1} = \mathbb{C}\times\mathbb{C}^*$, $\quad U_{\sigma_2} = \mathbb{C}^*\times\mathbb{C}$

$\downarrow$

$\mathbb{C}[x, y, y^{-1}]$

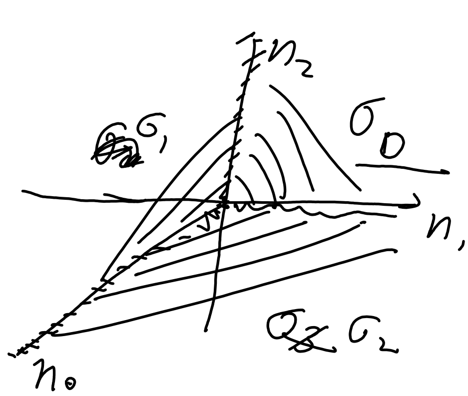
$U_{\{0\}} = \mathbb{C}^*\times\mathbb{C}^*$

$\downarrow$

$\mathbb{C}[x, x^{-1}, y, y^{-1}]$

$\to T_N\,\mathrm{emb}(\Delta) = \mathbb{C}^2 - \{(0,0)\}$

3. $n_0 := -n_1 - n_2$.



$\sigma_0 := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$

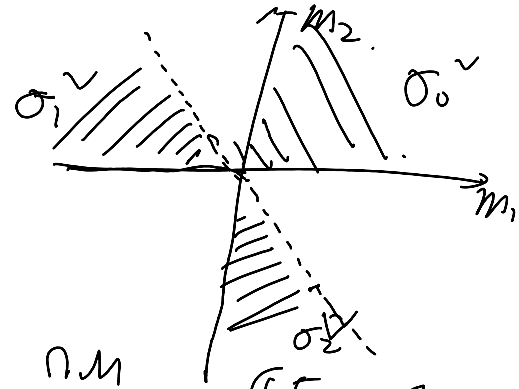
σ_1 ~~σ_2~~ $:= \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2$

σ_2 ~~σ_3~~ $:= \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1$

$\sigma_0 \cap \sigma_1 = \mathbb{R}_{\geq 0} n_2$

$\sigma_1 \cap \sigma_2 = \mathbb{R}_{\geq 0} n_0$

$\sigma_2 \cap \sigma_0 = \mathbb{R}_{\geq 0} n_1$



$\mathbb{C}^2$

$\sigma_0^\vee = \mathbb{R}_{\geq 0} m_1 + \mathbb{R}_{\geq 0} m_2 \xrightarrow{\cap M} \mathbb{C}[x, y]$

$\sigma_1^\vee = \mathbb{R}_{\geq 0}(-m_1) + \mathbb{R}_{\geq 0}(-m_1 + m_2) \longrightarrow \mathbb{C}[x^{-1}, x^{-1}y]$

$\sigma_2^\vee = \mathbb{R}_{\geq 0}(-m_2) + \mathbb{R}_{\geq 0}(m_1 - m_2) \longrightarrow \mathbb{C}[y^{-1}, xy^{-1}]$

$(\sigma_0 \cap \sigma_1)^\vee = \sigma_0^\vee + \sigma_1^\vee = \mathbb{R}(m_1) + \mathbb{R}_{\geq 0}(m_2) \longrightarrow \mathbb{C}[x, x^{-1}, y]$

$(\sigma_0 \cap \sigma_2)^\vee = \sigma_0^\vee + \sigma_2^\vee = \mathbb{R}(m_2) + \mathbb{R}_{\geq 0}(m_1) \longrightarrow \mathbb{C}[x, y, y^{-1}]$

$(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee = \mathbb{R}(-m_1 + m_2) + \mathbb{R}_{\geq 0}(-m_1) + \mathbb{R}_{\geq 0}(-m_2)$

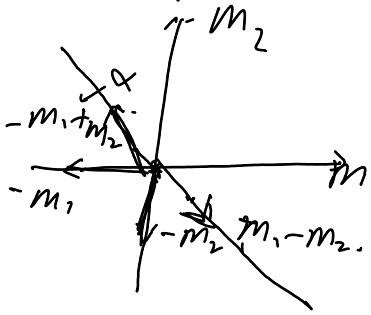
$\longrightarrow \mathbb{C}[x^{-1}, y^{-1}, x^{-1}y, xy^{-1}]$

redun

$[z_0 : z_1 : z_2]$

$\frac{z_1}{z_0} = e(m_1) \quad \frac{z_2}{z_0} = e(m_2)$

$\mathrm{Toremb}(\Delta) = \mathbb{P}^2_{\mathbb{C}}$



$(-m_1 + m_2) + (-m_2)$

$= (-m_1)$

General case (David Cox)

Def Affine toric variety

irreducible, affine variety V

$\underbrace{T_N \cong (\mathbb{C}^*)^n}_{\text{Zariski open}} \subseteq V$.

$T_N \curvearrowright T_N \Rightarrow T_N \curvearrowright V$.

eg: Nonnormal $C = V(x^3 - y^2) \subseteq \mathbb{C}^2$

Def: $A = \{m_1, \dots, m_s\} \subset M$, $\underline{Y_A}$.

$\Phi_A: T_N \longrightarrow \mathbb{C}^n$.

$t \longmapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$

$Y_A = \overline{\operatorname{im}\Phi_A}$

$\chi, e \leftrightarrow y_n$.

Prop $\underline{Y_A}$ $\underline{\mathbb{Z}A} \subseteq M$ sublattice

Linear algebra group.

character lattice $\mathbb{Z}A$,

$\dim Y_A = \operatorname{rank} \mathbb{Z}A$. T_N. ~~diag~~

Def Toric ideal I_A.

$\hat{\Phi}_A: \mathbb{Z}^s \longrightarrow M = N^\vee$

$\{e_j\} \longrightarrow \{m_j\}$

$\ker \hat{\Phi}_A = L \Rightarrow 0 \to L \to \mathbb{Z}^s \to M$

$(l_1, \dots, l_s) = l \to \begin{cases} l_+ = \sum_{l_i > 0} l_i e_i \\ l_- = -\sum_{l_i < 0} l_i e_i \end{cases} \Rightarrow l = l_+ - l_-$ $\to (\mathbb{Z}_{\geq 0})^s$

$\{x^{l_+} - x^{l_-} : l \in L\} \xrightarrow{\phi_A}$ vanish.

Prop $I(Y_A) = \langle x^\alpha - x^\beta : \alpha, \beta \in (\mathbb{Z}_{\geq 0})^s, \alpha - \beta \in L \rangle$

$= I_L$

lexicograph index

Def. $L \subseteq \mathbb{Z}^s$

(a). I_L lattice idea.

(b) prime. $\Rightarrow$ toric idea

Prop $I \subseteq \mathbb{C}[x_1, \cdots, x_s]$ toric

$\Leftrightarrow$ prime & generated by binomials.

Def. Affine semigroup. $\sigma^\vee \cap M = S_\sigma$

S.

commutative, finitely generated, embedded in M.

Prop. $S \subseteq M$ affine semigroup

(a). $\mathbb{C}[S]$ is an integral domain and finitely generated as a $\mathbb{C}$-algebra.

(b) $\mathrm{Spec}(\mathbb{C}[S])$ affine toric variety.

character lattice. $\mathbb{Z}S$ ... if $S = \mathbb{Z}_{\geq 0}A$, $\mathcal{A} \overset{\text{fin}}{\subseteq} M$.

$\mathrm{Spec}\, \mathbb{C}(\mathbb{C}[S]) = Y_A$,

$O(2^{2^n})$

Grobner basis,

Lemma. $\mathcal{A} \subseteq \mathbb{C}[M]$

$$\mathcal{A} = \bigoplus_{\chi^m \in \mathcal{A}} \mathbb{C}[\chi^m].$$

Theorem. V be an affine variety. TFAE:

① V ____ by Cox.

② $V = Y_{\mathcal{A}}$, for finite set $\mathcal{A}$ in a lattice.

③ V is an affine variety defined by a toric ideal

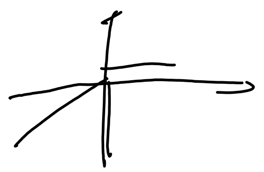
④ $V = \mathrm{Spec}(\mathbb{C}[S])$, for an affine semigroup S.

T_N-orbit ✓ amp div. base pt-free.

Div → Line Bundle? Proj

Oda. Ch2.

Fulton



tent.
2 dim.

T_N-orbit

$\forall \sigma \in \Delta$.

$$orb(\sigma) = \{u : M \cap \sigma^{\perp} \xrightarrow[\text{Group}]{} \mathbb{C}^*\}.$$

T_N orbit in $T_N emb(\Delta)$.

Every T_N-orbit is form of it.

$\Delta \longleftrightarrow$ orbits

$\sigma \longleftrightarrow orb(\sigma)$

① $\mathrm{orb}(\{0\}) = U_{\{0\}} = T_N$. $\dim(\sigma) = r$

② $\forall \sigma \in \Delta$, $\dim(\mathrm{orb}(\sigma)) = \mathrm{codim}(\sigma)$.

③ $\forall \sigma, \tau \in \Delta$, $\tau < \sigma \Longleftrightarrow \mathrm{orb}(\sigma) \subseteq \overline{\mathrm{orb}(\tau)}$.

④ $\forall \sigma \in \Delta$, $\exists!\ \mathrm{orb}(\sigma) \subset U_\sigma$, T_N-orbit closed.

$$U_\sigma = \bigsqcup_{\tau < \sigma} \mathrm{orb}(\tau)$$

⑤ $n \in N$, $\sigma \in \Delta$.

$$n \in \sigma \Longleftrightarrow \gamma_n,\ \underline{\lim_{\lambda \to 0} \gamma_n(\lambda)} \in U_\sigma.$$

$\updownarrow$

distinguished point

$$\gamma_\sigma := \begin{cases} 1 & \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$$

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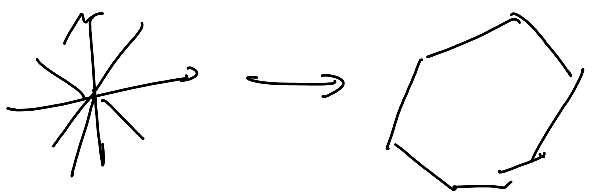
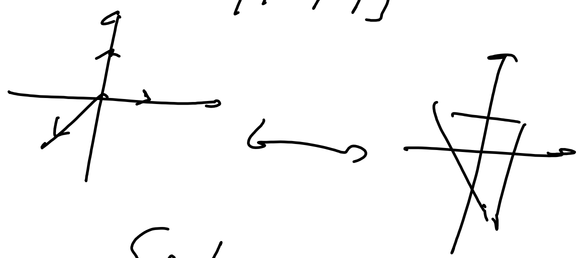
manifold with corner. Oda.

$T_N\mathrm{emb}(\Delta) / CT_N$ $CT_N = N \otimes U(1)$

$\downarrow$

$\{z \in \mathbb{C} \mid |z| = 1\}$





Galois Gauss Stoke