① Continue last time. + eg + Toric (e.g)

② General case of Toric variety. (brief without pf)

③ Orbit (start) 3 eg

David Cox → Aff : Aff Var $T_N$ ⊂ ; $T_N \times V \to V$ → × Normal

Thm. $\sigma$ ; S. C. R. P. C in $N_{\mathbb{R}}$.

$$S_\sigma = M \cap \sigma^\vee = \sum_{i=1}^{p} \mathbb{Z}_{\geq 0} m_i$$

$$U_\sigma = \{ u : S_\sigma \to \mathbb{C} ;\ u(0) = 1,\ u(m+m') = u(m)u(m')\ \forall m, m' \in S_\sigma \}$$

↑ Affine toric variety $\boxed{\chi^m \leftrightarrow e(m)}$

$$(e(m_1), \cdots, e(m_p)) : U_\sigma \to \mathbb{C}^p$$

$\{m_i\} \in S_\sigma$ generator.

$e : M \to T_N \cong (\mathbb{C}^*)^s$, $m$

pf: $\varphi : R_1 = \mathbb{C}[x_1, \cdots, x_n] \to R_2 = \mathbb{C}[e(m_1), \cdots, e(m_p)]$

① $\ker \varphi$ : $\langle x_1^{\nu_1} \cdots x_p^{\nu_p} - x_1^{\nu_1'} \cdots x_p^{\nu_p'} \rangle$

$$\sum \nu_i m_i = \sum \nu_i' m_i$$

$\forall f \in \ker\varphi$, $f = \sum b(\nu_1, \cdots, \nu_p)\, x_1^{\nu_1} \cdots x_p^{\nu_p}$ $\quad b : \mathbb{Z}^p \xrightarrow[\text{Set}]{} \mathbb{C}$ only finite none zero

$$0 = \varphi(f) = \sum b(\nu_1, \cdots, \nu_p)\, e(\nu_1 m_1 + \cdots + \nu_p m_p)$$
$$= \sum_{m \in S_\sigma} \Big( \sum_{\sum \nu_i m_i = m} b(\nu_1, \cdots, \nu_p) \Big) e(m)$$

$$\Rightarrow \forall m \in S_\sigma, \quad \sum_{\sum \nu_i m_i = m} b(\nu_1, \cdots, \nu_p) = 0.$$

$\sum_{i=1}^{N} x_i = 0$ Linear equation.

Basis of solution $\{x_1 - x_i\}_{i=1}^{\cdots}$

Gauss Algori step by step

$c\ \{x_i\}$ fin none

② Intergal domain

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_p^{\pm 1}]$$ Integral domain.

$\uparrow$ subring

$\mathbb{C}[S_\sigma]$ も co <u>Integral domain.</u>

③. Dim. $\underline{\dim \mathbb{C}[S_\sigma]} = \mathrm{tr\,deg}_{\mathbb{C}}\, \mathbb{C}\, \underbrace{\mathrm{Frac}(\mathbb{C}[S_\sigma])}$

$$\mathrm{Frac}(\mathbb{C}[S_\sigma]) = \mathrm{Frac}(\mathbb{C}[M]) = \mathrm{Frac}(\mathbb{C}[x_1, \ldots, x_p])$$

④ <u>Normal</u> $\sigma = \sum_{i=1} \mathbb{R}_{\geq 0} n_i$

$$\sigma^\vee = \bigcap (\underbrace{\mathbb{R}_{\geq 0} u_i}) ^\vee = \underbrace{\bigcap (H^+(u_i; 0))}$$

$$S_\sigma = \sigma^\vee \cap M = (\ \cdots\ ) \leftarrow \underline{\cap M}.$$

$$\downarrow$$

$$\mathbb{C}[S_\sigma] = \bigcap_{i=1}^{r} \mathbb{C}[S_{\sigma_i}], \quad S_{\sigma_i} = H^+(u_i; 0) \cap M$$

: Normal $\cap$ Normal $\Rightarrow$ <u>Normal?</u> Ring <u>integral</u> closure

$\underline{S_{\sigma_i}}$ $u$. of $\sigma = \mathbb{R}_{\geq 0} n$. $\forall k > 1$, $\frac{u}{k} \notin N$

<u>σ</u> ray $\underbrace{\mathbb{C}[S_\sigma]} = \mathbb{C}[x_1, x_2^{\pm}, x_3^{\pm}, \ldots, x_p^{\pm}]$

$$= \underbrace{\mathbb{C}[x_1, \ldots, x_p]}_{\text{normal}} \underbrace{x_2 x_3 \cdots x_p}$$

eg 1. ① $\mathbb{Z}$-basis of $N = \{n_1, n_2\}$

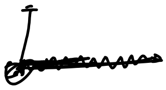
$M = \{m_1, m_2\}$

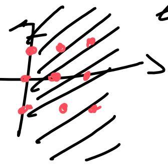
$\sigma = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 \longrightarrow$  $\mathbb{C}\times\mathbb{C}^*$

$\sigma^\vee = \mathbb{R}_{\geq 0} m_1 + \mathbb{R}_{\geq 0} m_2 \longrightarrow$  $\{u(m_2) \neq 0\} \cap U_\sigma$

$S_\sigma = \mathbb{Z}_{\geq 0} m_1 + \mathbb{Z}_{\geq 0} m_2 \qquad \mathbb{C}[S_\sigma] \cong \mathbb{C}[x, y]$

$\Rightarrow U_\sigma \cong \mathbb{C}^2$

② $\sigma = \mathbb{R}_{\geq 0} n_1 \longrightarrow$  red $S_\sigma = \mathbb{Z}_{\geq 0} m_1 + \underbrace{\mathbb{Z} m_2}_{\mathbb{Z}_{\geq 0} m_2 + \mathbb{Z}_{\geq 0}(-m_2)}$

$\sigma^\vee = \mathbb{R}_{\geq 0} m_1 \longrightarrow$ 

$\mathbb{C}\times\mathbb{C}^*$

$U_\sigma \longrightarrow \mathbb{C}^3$

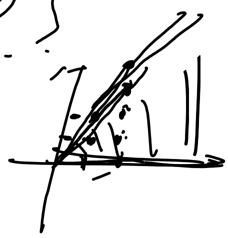
$u \longmapsto (e(m_1), e(m_2), e(-m_2)) \qquad 1 = u(0) = u(m_2 + (-m_2))$

$u(m_2) \neq 0. \qquad u(m_2) \to 0. \qquad = u(m_2) \cdot u(-m_2)$

$u \to \mathbb{C}. \qquad = 0 \cdot u(-m_2)$

③ $\sigma = \{0\} \qquad S_\sigma = M. \qquad \mathbb{C}^* \times \mathbb{C}^*$

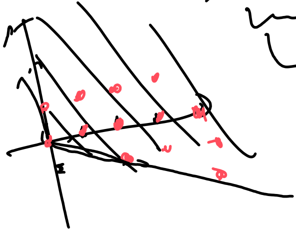
$\Rightarrow U_\sigma = \{(u_1, u_2) \in \mathbb{C}\times\mathbb{C} : u_1 u_2 \neq 0\}$



④ $\sigma = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0}(n_1 + 2n_2)$

$\sigma^\vee = \mathbb{R}_{\geq 0}(2m_1 - m_2) + \mathbb{R}_{\geq 0} m_2$ Hilbert basis

$S_\sigma = \mathbb{Z}_{\geq 0} m_1 + \mathbb{Z}_{\geq 0} m_2 + \mathbb{Z}_{\geq 0}(2m_1 - m_2)$ generators of $\sigma \cap M$. $\sigma^\vee \cap M$.



Hilbert basis ← np-hard

Algorith Elliot–Macmahon Algorithm

GPU

Prop $\sigma$ S.C.R.P.C. in $N_{\mathbb{R}}$. $\sigma^\vee$ is R.C. in $M_{\mathbb{R}}$.

If $\tau < \sigma$ (face), $\exists m_0 \in M \cap \sigma^\vee$.   Farkas Thm (Linear Programming)

$$\tau = \sigma \cap \{m_0\}^\perp = \{ y \in \sigma ; \langle m_0, y \rangle = 0 \}$$

$\tau \to$ ☆ in $N_{\mathbb{R}}$.   $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$

$$U_\tau = \{ u \in U_\sigma : u(m_0) \neq 0 \} \in U_\sigma. \text{ openset.}$$

pf: $m' \in \sigma^\vee$.  $\tau = \sigma \cap \{m'\}^\perp$.

① $\boxed{m' \in M}$     $C = \coprod_{F < C} \mathrm{RelInt}(F)$

1. $\sigma^\vee$ R...  2. face is relative inner of $\sigma$.

$a \in \mathbb{Z}_{\geq 0}$.  $am' \in M$.

② ☆ $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_0)$

$$(\sigma \cap \{m_0\}^\perp)^\vee = \sigma^\vee + (\{m_0\}^\perp)^\vee$$

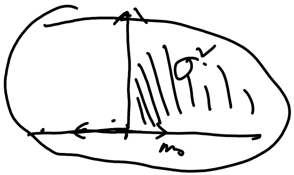
$\mathbb{R}_{\geq 0}(m_0) + \mathbb{R}_{\geq 0}(-m_0)$

Show ☆ $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$

- $\supseteq$. $S_\tau = M \cap \tau^\vee \supset M \cap \sigma^\vee + \mathbb{Z}_{\geq 0}(-m_0)$
  $= S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$
- $\subseteq$. $\tau^\vee, \sigma^\vee$  $\forall m \in S_\tau$ : $\exists a \in \mathbb{Z}_{\geq 0}$. $(m + a m_0) \in \sigma^\vee$
  $\Rightarrow M \cap \tau^\vee = S_\tau$   $m, m_0 \in M$.

$\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_0)$   $m_0 \in M$.

$S_\tau \supseteq S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$



$S_\tau \subseteq S_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$. $m_0 \,|\, \forall a, \in \sigma^\vee$

$\forall m \in S_\tau$, $\forall a \in \mathbb{Z}_{\geq 0}$. $m = m' - a m_0$.

$m + a m_0 \in \sigma^\vee$ $\sigma$

$\ni m$ $m' = m + a m_0$ $a \in \mathbb{Z}$. $m' \in \sigma^\vee$

$m_0$ $a m_0$

Thm. fan $\Delta$ in $N$, glue $\{U_\sigma : \sigma \in \Delta\}$

$\Rightarrow$ Hausdorff complex analytic space

$T_N\mathrm{emb}(\Delta) := \bigcup_{\sigma \in \Delta} U_\sigma$. $\leftarrow$ Toric variety or Torus embedding associated to the fan $(N, \Delta)$

irreducible, normal. $\dim = r = \operatorname{rank} N$.

pf: $\sigma \in \Delta$. $U_\sigma$. $r$-dim. irreducible. algebraic subset normal.

$\sigma_1, \sigma_2 \in \Delta$. $\sigma_1 \cap \sigma_2 \begin{cases} < \sigma_1 \\ < \sigma_2 \end{cases}$ glue $U_{\sigma_1}, U_{\sigma_2}$, along $U_{\sigma_1 \cap \sigma_2}$.

Show. Hausdorff (Separated in GAGA)

$X \to X \times X$ is closed.

$p \mapsto (p, p)$.

$\Rightarrow$ $\boxed{\begin{array}{ccc} U_{\sigma_1 \cap \sigma_2} & \longrightarrow & U_{\sigma_1} \times U_{\sigma_2} \\ u & \longmapsto & (u', u'') \end{array}}$ $u' = u|_{S_{\sigma_1}}$ $u'' = u|_{S_{\sigma_2}}$.

1. Show $S_{\sigma_1 \cap \sigma_2} = S_{\sigma_1} + S_{\sigma_2}$.

$(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee \Rightarrow S_{\sigma_1 \cap \sigma_2} \supset S_{\sigma_1} + S_{\sigma_2}$.

$\sigma_1 \,|\, \sigma_2$ $\in \{m_0\}^\perp$

$\exists m_0 \in M$, $\sigma_1, \sigma_2$ separated by $\{m_0\}^\perp$.

$\Rightarrow m_0 \in S_{\sigma_1}$, $-m_0 \in S_{\sigma_2}$.

$\sigma_1 \cap \sigma_2 = \sigma_1 \cap \{m_0\}^{\perp} = \sigma_2 \cap \{m_0\}^{\perp}$.

$\Rightarrow S_{\sigma_1 \cap \sigma_2} = S_{\sigma_1} + \mathbb{Z}_{\geq 0}(-m_0) \subset S_{\sigma_1} + S_{\sigma_2}$.

$S_{\sigma_1}$, Generators $\to \{m_1, \dots, m_p\}$

$S_{\sigma_2}$ $\to \{m_1', \dots, m_q'\}$

$U_{\sigma_1 \cap \sigma_2} \longrightarrow \mathbb{C}^{p+q}$

$(e(m_1), \dots, e(m_p), e(m_1'), \dots, e(m_q'))$

$U_{\sigma_1} \longrightarrow \mathbb{C}^{p}$ $\qquad$ $U_{\sigma_2} \longrightarrow \mathbb{C}^{q}$

$U_{\sigma_1 \cap \sigma_2} \in \mathbb{C}^{p+q}$ closed

$U_{\sigma_1} \in \mathbb{C}^{p}$ closed.

$U_{\sigma_2} \in \mathbb{C}^{q}$ closed.

$U_{\sigma_1 \cap \sigma_2} \in U_{\sigma_1} \times U_{\sigma_2}$.

$\square$

$U_{\sigma_1} \cap U_{\sigma_2} \sim U_{\sigma_1 \cap \sigma_2}$

$U_{\sigma_1 \cap \sigma_2} \to U_{\sigma_1} \times U_{\sigma_2}$.