$\forall m_1, m_2 \in M$,

$\mathbb{e}(m_1 + m_2) = \mathbb{e}(m_1)\,\mathbb{e}(m_2)$,

$\mathbb{e}(0) = 1$

$\mathbb{Z}$-basis of N; $\{n_1, \cdots, n_r\}$,

$\cdots \quad - \quad M \quad \{m_1, \cdots, m_r\}$

$$u_j = \mathbb{e}(m_j)$$

$$T_N \xrightarrow{\sim} (\mathbb{C}^*)^r$$

$$t \longmapsto (\underbrace{u_1(t), \cdots, u_r(t)}_{u_i \text{ coordinate for } T_N})$$

$\underline{m = \sum a_i m_i}$,

$$\mathbb{e}(m) = \underbrace{\prod u_i^{a_i}}_{\text{Laurent monomial } T_N},$$

$\forall n \in N$, one-parameter subgroup

$$\gamma_n : \mathbb{C}^* \longrightarrow T_N,$$

$$\gamma_n(\lambda)(m) = \lambda^{\langle m, n \rangle}, \ \forall \lambda \in \mathbb{C}^*, \ m \in M.$$

$n = \sum b_j n_j$

$$\gamma_n : \mathbb{C}^* \longrightarrow T_N,$$
$$\lambda \longmapsto (\lambda^{b_1}, \lambda^{b_2}, \cdots, \lambda^{b_r}) \in (\mathbb{C}^*)^r$$

Prop. $S_\sigma = M \cap \sigma^\vee = \sum_{fin} \mathbb{Z}_{\geq 0} m_i$

$\underset{\sim}{\text{s.c.r.p.c.}}$ $\sigma \leq N_{\mathbb{R}}$. Let

$U_\sigma := \{ u : S_\sigma \to \mathbb{C} ;\ u(0) = 1,$

$\underset{\sim}{\text{Affine Toric}}$ $u(m+m') = u(m) u(m'), m, m' \in S_\sigma \}$

$e(m)(u) := u(m), \ \forall m \in S_\sigma, u \in U_\sigma.$

$$\underset{\sim}{(e(m_1), \cdots, e(m_p))} :$$
$$U_\sigma \longrightarrow \mathbb{C}^p = \mathbb{C} \times \cdots \times \mathbb{C}$$

1-1

U_σ is algebraic subset of $\mathbb{C}^p$ defined as the set of solution of a system of equations

(monomial) = (monomial)

(toric ideal)

r-dim, irreducible, normal, complex, analysis space on U_σ

$\{m_1, \cdots, m_\sigma\}$

Each $m \in S_\sigma$, polynomial $e(m)$ on U_σ, holomorphism

Rmk.

$\mathbb{C}[M] := \bigoplus_{m \in M} \mathbb{C}e(m)$. group alg of M over $\mathbb{C}$.

$$\Big(\sum_i a_i e(m_i)\Big)\Big(\sum_j (b_j e(m_j))\Big) = \sum_{i,j} a_i b_j e(m_i + m_j)$$

Fulton $\chi^m \chi^{m'} = \chi^{m+m'}$

Affine scheme $\mathrm{Spec}(\mathbb{C}[M])$

$\underline{\mathbb{C}\text{-pts}}$ $\quad \underline{\mathbb{C}[M]} \longrightarrow \underline{\mathbb{C}}$

Thm: $\underline{U(S_\sigma) \neq 0}$

pf: $m_1, \cdots, m_p \in S_\sigma$ gen.

$u \in U_\sigma$, $u(m_j) = e(m_j)(u) \in \mathbb{C}$, $\forall 1 \leq j \leq p$.

$\underline{a = (a_1, \cdots, a_p) \in \mathbb{C}^p}$

$u \in U_\sigma$, $u(m_j) = a_j$

$\underline{S_\sigma = M \cap \sigma^\vee}$

$\mathbb{C}[\vec{x}] = \mathbb{C}[x_1, \cdots, x_p]$

$\longrightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[e(m_1), \cdots, e(m_p)]$

$x_j \mapsto e(m_j)$ $\forall j$.

$\ker = \langle f_1, \cdots, f_q \rangle$, $\exists u \in U_\sigma$, $u(m_j) = a_j$

$\Leftrightarrow$ $\underline{f_1(a) = f_2(a) = \cdots = f_q(a) = 0}$

$\forall f \in I,$ $\qquad (v_1, \cdots, v_p) \in \mathbb{Z}$

$0 = f = \sum b(v_1, \cdots, v_p)\, x_1^{v_1} \cdots x_p^{v_p}$

$\mathbb{C}$ $\qquad 0$

$$= \sum b(v_1, \cdots, v_p)\, e(v_1 m_1 + \cdots + v_p m_p)$$

$$= \sum_{m \in S_0}' \left(\sum'' b(v_1, \cdots, v_p) \right) e(m)$$

$v_1 m_1 + \cdots + v_p m_p = m$

$y_1 - y_5$

~~0~~ $y_1 - y_2$

$y_1 - y_3$

$$\left[b(\vec{v}_1(m)), \cdots b(\vec{v}_p(m)) \right] \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = 0$$

$$X_1^{v_1} X_2^{v_2} \cdots X_p^{v_p} - X_1^{u_1} X_2^{u_2} \cdots X_p^{u_p}$$

$\mathbb{C}[S_0]$ of U_0 as an affine alg variety