Jan, 30, 2024.

Polytope & T_N-Divisor.

$P \subseteq M_{\mathbb{R}}$ full dim, latt p. , $\dim(P) = n$.
$\overset{\shortparallel}{\mathbb{R}^n}$

$P = \{ m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq \underline{-a_F},\ \forall F \in P.\text{Facet} \}$.
$\in \mathbb{Z}$. $\underline{u_F} \in N$. M^*

Normal fan Σ_P σ_Q. $Q \preceq P$. where
$\sigma_Q = \text{Cone}(u_F : F \text{ consists } Q)$

Prop. $P \subseteq M_{\mathbb{R}}$ full dim. latt.p. of dim n. and consider σ_Q in normal fan Σ_P of P. Then:
(a) $\dim Q + \dim \sigma_Q = n$, $\forall Q \preceq P$.
(b). $N_{\mathbb{R}} = \bigcup_{v \in P.\text{vertex}} \sigma_v = \bigcup_{\sigma_Q \in \Sigma_P} \sigma_Q$.

$\Rightarrow \Sigma_P$ complete.

$\begin{cases} P.\text{Vertex}^{0} \longleftrightarrow \Sigma_P(n) \\ P.\underbrace{\text{Facets}^{n-1}}_{F} \longleftrightarrow \underbrace{\Sigma_P(1)}_{\rho} \end{cases}$

X_P. $D_P = \sum a_F D_F$

Prop D_P is $\underline{\text{CDiv}}$ on X_P and $\underline{D_P \neq 0}$.

pf: Vertex $\longleftrightarrow \Sigma(n)$
$v \longleftrightarrow \sigma_v$.

ray $\rho_F \in \sigma_v(1)$ iff $\underline{v \in F}$.
$\Rightarrow \langle v, \underline{u_F} \rangle = -a_F$. P is latt. p. $\Rightarrow v \in M$.

We have $v \in M$, s.t.

$\langle v, u_F \rangle = -a_F$. $\forall \rho_F \in \sigma_v(1)$.

So D_P is C Div. $\rightarrow$ LHS $\sum_\rho (a_\rho + \langle m, u_\rho \rangle) D_\rho$.

$$\begin{cases} D_P + \underline{\operatorname{div}(\chi^m)} = D_{P-m} \\ \underline{D_P \sim 0}. \quad \text{normal } \Sigma_P \text{ is complete.} \end{cases}$$

$\square$

RMK:

$m_{\sigma_v} \longleftrightarrow v$.

$$\begin{cases} \{m_{\sigma_v}\}_{\sigma_v \in \Sigma_P(0)} = \{v \mid v \text{ is vertex of } P\} \\ [D_P] \in \operatorname{Pic}(X_P). \quad D = D_P + \operatorname{div}(\chi^m). \quad \underline{D_{P-m}}. \\ X_P = X_{P+m}. \end{cases}$$

The sheaf of T_N-invariant Div.

eg: $\mathbb{P}^n$, $D_0, \ldots, D_n$, $u_0, \ldots, u_n$.

$Cl(\mathbb{P}^n) \cong \mathbb{Z}$, $D_0 \sim D_1 \sim \cdots \sim D_n$.

$\mathcal{O}_{\mathbb{P}^n}(D_0) \simeq \cdots \simeq \mathcal{O}_{\mathbb{P}^n}(D_n) \longleftrightarrow \mathcal{O}_{\mathbb{P}^n}(1)$.

$\mathcal{O}_{\mathbb{P}^n}(kD_0)$ ---------- $\longleftrightarrow \mathcal{O}_{\mathbb{P}^n}(k)$

Global section.

D, X_Σ, $\underline{\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))}$

$\Vert$

Prop Weil Div,

$$\boxed{\bigoplus_{\operatorname{div}(\chi^m)+D \geq 0} \mathbb{C} \cdot \chi^m}$$

pf:

If $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$, $D + \mathrm{div}(f) \geq 0$.

$\Rightarrow$ $\mathrm{div}(f)|_{T_N} \geq 0$, since $D|_{T_N} = 0$.

$\mathbb{C}[M]$ of T_N $\Rightarrow$ $f \in \mathbb{C}[M]$.

$$\underbrace{\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))}_{T_N\text{-invariant}} \subseteq \mathbb{C}[M]$$

T_N-action:

$$= \bigoplus_{\chi^m \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))} \mathbb{C}\cdot\chi^m.$$

since $\chi^m \in$? $\Leftrightarrow$ $\mathrm{div}(\chi^m) + D \geq 0$. $\square$

The polyhedron of Divisor.

For $D = \sum a_\rho D_\rho$, $m \in M$. $\mathrm{div}(\chi^m) + D \geq 0$.

$\sim$ $\langle m, u_\rho\rangle + a_\rho \geq 0$ $\forall \rho \in \Sigma(1)$

$\geq -a_\rho$

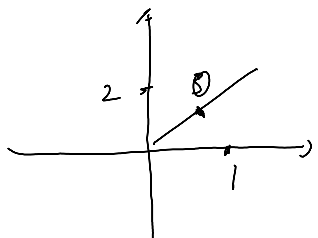
$P_D = \{ m \in M : \langle m, u_\rho\rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\}$.

eg: $Bl_0(\mathbb{C}^2)$, $\mathbb{C}^2$ at 0.

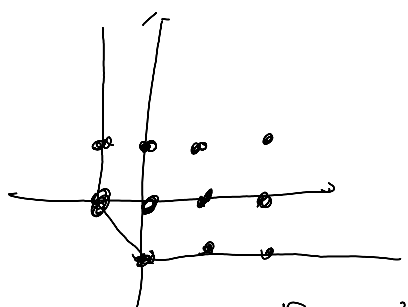
$u_0 = e_1 + e_2$ $\quad u_1 = e_1$ $\quad u_2 = e_2$.

D_0 $\quad D_1$ $\quad D_2$.

$D = D_0 + D_1 + D_2$.



$\longrightarrow$



P_D is not bounded.

eg: $\mathcal{H}_2$. $u_1 = -e_1 + 2e_2$, $u_2 = e_2$, $u_3 = e_1$, $u_4 = -e_2$.

Σ_2 D_1 D_2 D_3 D_4

$(D_1, D_2$ basis $Cl(\mathcal{H}_2) \cong \mathbb{Z}^2)$

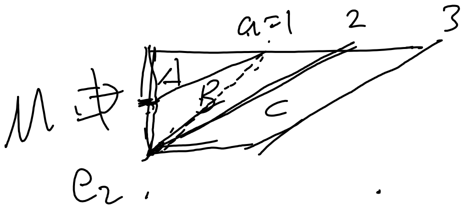
$aD_1 + D_2, a \in \mathbb{Z}$, $P_a \subseteq \mathbb{R}^2$.

for a.

$\Sigma_2 \times \mathcal{H}_2$

$a=2$

$a=1$.



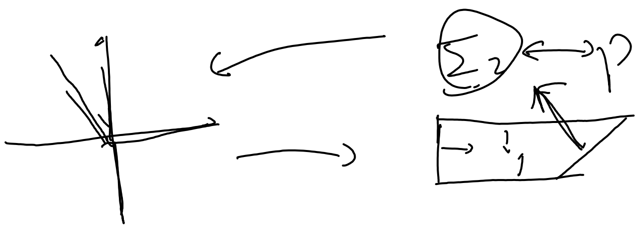
$P_1 = A$

$A \cup B$

$A \cup B \cup C$.

$a=2$. $\times$ normal.

$a=3$. normal Σ_2. □



eg: $\mathcal{O}_{\mathbb{P}^n}(k)$, $D = kD_0$.

$$P_D = \begin{cases} \emptyset & k<0 \\ k\Delta_n, & k \geq 0. \end{cases}$$ standard n-simplex

Laurent monomials $t^m = \prod_{i=1}^n t_i^{a_i}$ $m = (a_1, \dots, a_n)$

$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong \{ f \in \mathbb{C}[t_1, \dots, t_n] : \deg(f) \leq k \}$. affine

$F = x_0^k f(x_1/x_0, \dots, x_n/x_0) \in \mathbb{C}[x_0 : \dots : x_n]$.

$\cong$ $\{ F \in \mathbb{C}[x_0 : \dots : x_n] ; \deg(F) = k \}$ □.

eg: X_P, P full dim. lac $P \subseteq M_{\mathbb{R}}$.

facet of P $\longleftrightarrow$ CDiv. $P_{D_P} = P$.

$$\Gamma(X_P, \mathcal{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m.$$

$\chi^m \Rightarrow X_{P \cap M}$.

$kD_P \xrightarrow{\text{gives}}$ polytope. kP.

$\Gamma(X_P, \mathcal{O}_{X_P}(kD_P)) = \bigoplus_{m \in (kP) \cap M} \mathbb{C} \cdot \chi^m$. $\quad$ n. $k \geq n-1$

X_P, by χ^m comes from. $\mathcal{O}_{X_P}(D_P)$ $\quad$ $X_{(kP) \cap M} = X_P$.

$\dim \Gamma(X_P, \mathcal{O}_{X_P}(D_P)) = |P \cap M|$ $\quad$ □.

RMK:

① $P_{D_P} = P$ $\quad$ $P \xrightarrow{F \leftrightarrow \rho} D_P \longrightarrow P_{D_P}$.

u_F, a_F. $\quad$ $\{(u_\rho, a_\rho)\}$

, $D \mapsto P_D$.

- $P_{kD} = kP_D$ $\quad$ $\langle \cdot, u_\rho \rangle \geq -a_\rho$
- $P_{D + \mathrm{div}(\chi^m)} = P_D - m$.
- $P_D + P_E \subseteq P_{D+E}$. not ample. $\quad$ $P_D + P_D \subseteq 2P_D = P_{2D}$?

exception.

(ney) $\quad$ D, E eff. $\quad$ $D+E$.

□

Basepoint Free Divisor.

X_Σ of Σ. in $N_{\mathbb{R}} \cong \mathbb{R}^n$, $D = \sum a_\rho D_\rho$ $\quad$ CDiv.

Prop. $\Sigma(n)$. TFAE:

(a) D has no basepoint. i.e., $\mathcal{O}_{X_\Sigma}(D)$ is generated by global section.

(b) $m_\sigma \in P_D$ $\forall \sigma \in \Sigma(n)$.

$\hookrightarrow$ CData.

pf:

D is generated by global section and $\sigma \in \Sigma(n)$.

The T_N-orbit $\sim \sigma$ is p. (by Orbit-Cone Corresp...)

$\{p\} = \underbrace{\bigcap_{\rho \in \sigma(1)} D_\rho}$,

$\exists\, s \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$, $p \notin \text{supp div}_0(s)$.

$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ spanned by χ^m; $m \in P \cap M$. assume $s = \chi^m$.

$$\text{div}_0(s) = D + \text{div}(\chi^m) = \sum_\rho (a_\rho + \langle m, u_\rho \rangle) D_\rho$$

$\Rightarrow a_\rho + \langle m, u_\rho \rangle = 0$, $\forall \rho$

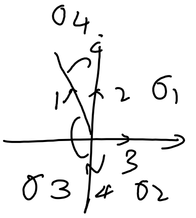
$\dim(\sigma) = n$. $\Rightarrow$ $m_\sigma = m \in P_D$.

For converse, $\sigma \in \Sigma(n)$. Some $m_\sigma \in P_D$.

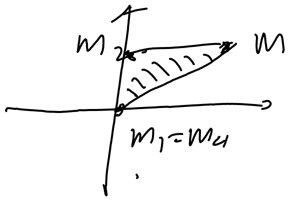
$\chi^{m_\sigma} \Rightarrow s$. $\text{supp div}_0(s) \cap U_\sigma = \emptyset$. $\Rightarrow s|_{U_\sigma} \neq 0$.

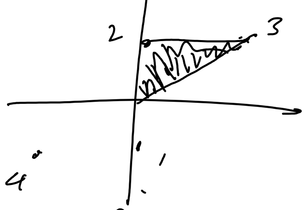
U_σ cover X_Σ. $\square$.

eg: $\mathcal{H}_2$.



Σ_2. $D = D_4$, $D' = D_2 + D_4$





P_D ✓. $P_{D'}$ ✗. $\square$

Support Function & it's graph.

$D = \sum a_\rho D_\rho$ CDiv. X_Σ, $\varphi_D : |\Sigma| \longrightarrow \mathbb{R}$.

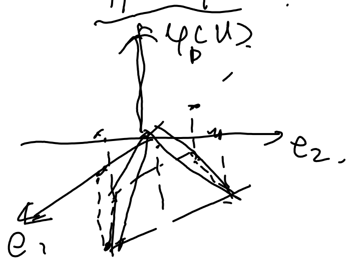
- φ_D linear on each cone σ,
- $\varphi_D(u_\rho) = -a_\rho$. $\forall \rho \in \Sigma(1)$

(Data. $\{m_\sigma\}_{\sigma \in \Sigma} \Rightarrow \varphi_D|_\sigma(u) = \langle m_\sigma, u\rangle \quad \forall u \in \sigma.$

"σ runs through all cones in Σ".

Eg: $\mathbb{P}^1 \times \mathbb{P}^1$. $D = D_1 + D_2 + D_3 + D_4$. $\varphi_D(u_i) = -1$.

tent.



Convex Function.

Def $S \subseteq N_{\mathbb{R}}$ be convex. $\varphi: S \to \mathbb{R}$ is convex if

$$\varphi(tu + (1-t)v) \geq t\varphi(u) + (1-t)\varphi(v)$$

$u, v \in S$, $t \in [0,1]$.

Full Dim Convex Support. (on complete fan).

$$\begin{cases} |\Sigma| \subseteq N_{\mathbb{R}} \\ \dim|\Sigma| = n = \dim N_{\mathbb{R}} \end{cases}$$

Σ convex support of full dimension, such fans satisfy:

$$|\Sigma| = \mathrm{Cone}(u_\rho : \rho \in \Sigma(1)) = \bigcup_{\sigma \in \Sigma(n)} \sigma.$$

Supp Func & convexity.

Σ, $N_{\mathbb{R}} \simeq \mathbb{R}^n$. τ is wall. $\tau = \underbrace{\sigma \cap \sigma'}_{\in \Sigma(n)} \in \Sigma(n-1)$

Lemma. Let D CDiv on X_Σ, Σ convex support full dim.. TFAE:

(a). supp func $\varphi_D : |\Sigma| \to \mathbb{R}$ is convex (b). $\varphi_D(u) \le \langle m_\sigma, u\rangle \quad \forall u \in |\Sigma|,\ \sigma \in \Sigma(n)$.

(c). $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u\rangle \quad \forall u \in |\Sigma|$.

(d). $\forall$ wall $\tau = \sigma \cap \sigma'$, $\exists u_0 \in \sigma' \setminus \sigma$, with $\varphi_D(u_0) \le \langle m_\sigma, u_0\rangle$.

pf: Assume (a). fix $v \in \mathrm{int}(\sigma)$, $\sigma \in \Sigma(n)$. Given $u \in |\Sigma|$, $t \in (0,1)$. $tu + (1-t)v \in \sigma$.

$$\langle m_\sigma, tu + (1-t)v\rangle = \varphi_D(tu + (1-t)v)$$
$$\ge t\varphi_D(u) + (1-t)\varphi_D(v)$$
$$= t\varphi_D(u) + (1-t)\langle m_\sigma, v\rangle$$

$\langle m_\sigma, u\rangle \ge \varphi_D(u)$. proved (b)

Since $\varphi_D(u) = \langle m_\sigma, u\rangle$, $u \in \sigma$, (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) Basic prop

$\min_{fin} \{ f : f \text{ is linear} \}$ is also convex.

(b) $\Rightarrow$ (d) obvious show (d) $\Rightarrow$ (b)

Assume (d) ✓, fix $\tau = \sigma \cap \sigma'$,

$\langle m_{\sigma'}, u\rangle \le \langle m_\sigma, u\rangle$ u, σ' on the same side of τ.

The wall is defined by

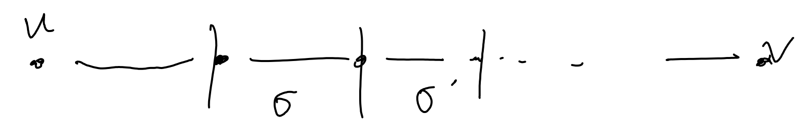
$$\langle m_{\sigma'} - m_\sigma, u\rangle = 0.$$

(d) $\Leftrightarrow$ (halfspace) $\underline{\qquad} \ge 0$.

containing σ'.

$u \in |\Sigma|$ convex & $\sigma \in \Sigma(n)$ ... $v \in \mathrm{int}(\sigma)$.

$\overrightarrow{uv}$ segment



$\langle m_\sigma, u\rangle \geq \langle m_{\sigma'}, u\rangle \geq \cdots$

$\varphi_D(u) \leq \langle m_\sigma, u\rangle$. (b) ✓ $\square$.

Lemma. Σ, $D = \sum a_\rho D_\rho$ be CDiv X_Σ.

$$P_D = \{ m \in M_{\mathbb{R}} : \varphi_D(u) \leq \langle m, u\rangle, \ \forall u \in |\Sigma| \}$$

pf: $\varphi_D(u) \leq \langle m, u\rangle$ $\forall u \in |\Sigma|$. Apply $u = u_\rho$

$$-a_\rho = \varphi_D(u_\rho) \leq \langle m, u_\rho\rangle \Rightarrow m \in P_D$$

Opposite inclusion,

$m \in P_D$ $u \in |\Sigma|$ $\Rightarrow u \in \sigma \in \Sigma$.

$\Rightarrow$ $u = \sum_{\rho \in \sigma(1)} \lambda_\rho u_\rho$, $\lambda_\rho \geq 0$. Then

$$\langle m, u\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m, u_\rho\rangle \underset{m \in P_D}{\geq} \sum_{\rho \in \sigma(1)} \lambda_\rho(-a_\rho)$$

$$= \underbrace{\sum_{\rho\in\sigma(1)} \lambda_\rho \varphi_D(u_\rho) = \varphi_D(u)}_{\text{by def of } \varphi_D} \quad \square$$

Thm. $|\Sigma|$ conv. full. dim.
φ_D supp fun. of D on X_Σ. TFAE:

(a). D is basepoint free.

(b). $m_\sigma \in P_D$ $\forall \sigma \in \Sigma(n)$

(c) $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u\rangle$ $u \in |\Sigma|$.

(d). $\varphi_D : |\Sigma| \to \mathbb{R}$ is convex.

If Σ complete:

(e). $P_D = \mathrm{Conv}(m_\sigma : \sigma \in \Sigma(n))$

(f). $\{m_\sigma \mid \sigma \in \Sigma(n)\}$ set of vertices of P_D.

(g). $\varphi_D(u) = \min_{m \in P_D} \langle m, u \rangle$. $\forall u \in N_{\mathbb{R}}$.

pf:

φ_D convex $\Leftrightarrow \varphi_D(u) \le \langle m_\sigma, u \rangle$, $\forall \sigma \in \Sigma(n)$, $u \in |\Sigma|$.

$\Leftrightarrow m_\sigma \in P_D \quad \forall \sigma \in \Sigma(n)$

(a), (b), (c), (d),

?

Assume (b) $\quad$ P_D polytope, since Σ complete.

$m_\sigma \in P_D$. $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle$

we obtain.

$$\varphi_D(u) \le \min_{m \in P_D} \langle m, u \rangle \le \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle = \varphi_D(u)$$

prove (g).

(g) $\Rightarrow$ (d), $\min_{\text{compact}} \{ f : f \text{ is linear} \}$ is convex.

(a), (b), (c), (d), (g).

(b) $\Rightarrow$ (f).

$\sigma \in \Sigma(n)$, $u \in \mathrm{int}(\sigma)$ $\quad a = \varphi_D(u)$

$$H_{u,a} = \{ m \in M_{\mathbb{R}} : \langle m, u \rangle = a \}$$

supporting hyperplane of P_D and.

$$H_{u,a} \cap P_D = \{ m_\sigma \}$$

This m_σ is vertex of P_D.

Conversely. Let $H_{u,a}$ be a supporting hyperplane of $v \in P_D$.

$\langle m, u \rangle \geq a$, $\forall m \in P_D$

$=$ iff $m = v$.

Since (b) ✓ $\longrightarrow$ (c)

$\longrightarrow$ (g). $\longrightarrow \varphi_D(u) = \min_{m \in P_D} \langle m, u \rangle = \langle m, v \rangle = a$.

we obtain,

$$\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle = a.$$

Hence, $\langle m_\sigma, u \rangle = a$ $\quad \sigma \in \Sigma(n)$

$$v \longleftrightarrow m_\sigma.$$

$\square$

Rmk: process. Toric

Newton polyhedron.

zero point equations

toric degeneration. trick on degree/index?

Grobner basis

$x_1^{a_1} \cdots x_n^{a_n} \longrightarrow M$.