1. New concepts for general case

2. Toric variety

3. General case

1. $V \cong \mathbb{R}^n$

$S \subset V \qquad V^\vee$

$$\begin{cases} S^\vee = \{u \in V^\vee ; \langle v, u \rangle \geq 0, \forall v \in S\} \\ S^\perp = \{ \cdots\cdots \langle v, u \rangle = 0 \cdots\cdots \end{cases}$$

Prop ① $S^\vee$ is a convex cone in $V^\vee$.

② $S^\perp$ is an $\mathbb{R}$-subspace of $V^\vee$.

③ $T \subseteq V^\vee$, ①, ②, $T^\vee \to V$, $T^\perp \to V$

④ $S^\perp \subseteq S^\vee$, $S^\vee \cap (-S^\vee) = S^\perp$

⑤ $S_1 \supset S_2 \Rightarrow (S_1^\vee \subset S_2^\vee)$, $(S_1^\perp \subset S_2^\perp)$

⑥ $(S_1 \cup S_2)^\vee = S_1^\vee \cap S_2^\vee$

$(S_1 \cup S_2)^\perp = S_1^\perp \cap S_2^\perp$

Thm (Farkas') Convex polyhedral cone C, $C^\vee$ is also convex polyhedral cone.

pf: V, $\dim V = n$

$$C = \sum_{i=1}^{N} \mathbb{R}_{\geq 0} v_i,$$

$$C^\vee = \bigcap_{i=1}^{N} H^+(v_i; 0)$$

half-space bound

$H^+(v_i; 0)$ n - linear independent vectors

$\{u_i^1, \cdots, u_i^n\}$ $\langle u_i^j, v_k \rangle \geq 0$, $k = 1, \cdots, N$

$\boxed{\{u_i^j\}_{i=1}^{N}} \rightarrow C^\vee$

$\langle u, v_i \rangle \geq 0$ □

Thm (Caratheodory's) $d = \dim(C)$.

$C = \sum_{i=1}^{N} \mathbb{R}_{\geq 0} v_i$, $\forall v \in C$,

$\exists \{a_{i_j}\}_{j=1}^{d} \subseteq \mathbb{R}_{\geq 0}$ other $a_i = 0$ $v = \sum_{i=1}^{N} a_i v_i$.

RelInt(C).

Def $\partial C = C \setminus \mathrm{RelInt}(C)$

Def $F \subset C \subset V$ face. $F < C$

if $F = C \cap \{u\}^{\perp}$, $u \in C^{\vee}$

F also a convex polyhedral cone

$\cap H^{+}(u, 0) \cap H^{+}(-u, 0)$

F is thus the intersection of C with the boundary $\partial H^{+}(u, 0)$

$\{u\}^{\perp}$

$$C = \bigsqcup_{F < C} \mathrm{RelInt}(F)$$

$\subseteq$

Lemma. $C \subset V$ convex polyhedral cone, $v \in C$;

1. $v \in \mathrm{RelInt}(C)$

$\Updownarrow$

2. $\langle v, u \rangle > 0$, $\forall u \in C^\vee \setminus C^\perp$.

$\Updownarrow$

3. $C^\vee \cap \{v\}^\perp = C^\perp$.

$\Updownarrow$

4. $C + \mathbb{R}_{\geq 0}(-v) = \mathbb{R}C := C + (-C)$

Def fan in N is a nonempty collection Δ of strongly rational polyhedral cone. satisfying the following conditions;

- $\forall F < \sigma \in \Delta, \Rightarrow F \in \Delta,$
- $\forall \sigma, \sigma' \in \Delta, \sigma \cap \sigma' < \sigma$
$< \sigma',$

$|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ is called support of Δ.

Prop σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$.

- $S_\sigma := \underline{M \cap \sigma^\vee}$. $0 \in S_\sigma$, $m, m' \in S_\sigma$
$\Rightarrow m + m' \in S$
- $S_\sigma := \sum_{\text{fin}} \mathbb{Z}_{\geq 0} m_i$.

* (Saturated), $\forall\, m \in M,\ a \in \mathbb{Z}^+,\ am \in S_\sigma,$
$\Rightarrow m \in S_\sigma$

* $S_\sigma + (-S_\sigma) = M,$) 4 points above.

Conversely, S of M $\exists\,\sigma$ above,
$S = S_\sigma$.

Rmk:

$$\begin{tikzcd}
V \arrow[r, leftrightarrow, "\text{dual}"] & V^* \\
\sigma \arrow[u] \arrow[r] \arrow[dr, leftrightarrow] & \sigma^\vee \arrow[u] \arrow[d, "\cdot \cap M"] \\
& S_\sigma
\end{tikzcd}$$

pf:

① , f.g.

② (Gordan) $\sigma^\vee$ is r-dim.

$$\sigma^\vee = \bigcup_{\text{fin}} \text{simplicial cone}$$

$\sigma^\vee$ simplicial cone

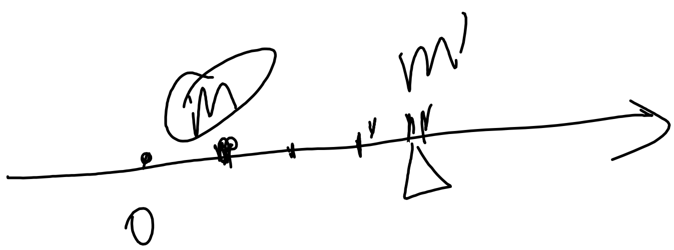
$$\sigma^\vee = \sum_{fin} \underline{\mathbb{R}_{\geq 0}}\, m_i \longrightarrow \mathbb{Z}_{\geq 0}$$

$\{m_i\} \subseteq S_\sigma$, $\mathbb{R}$-linearly independent.

$$M' = \sum_{fin} \mathbb{Z}_{\geq 0} m_i, \quad [M : M'] < +\infty$$

$\forall m \in \underline{S_\sigma}$, $\exists m' \in M' \cap \sigma^\vee$,

$$\underline{m - m'} = \underline{\sum a_i m_i,\ 0 \leq a_i < 1}$$



$\underline{m = \frac{1}{3} m'}$

- 4-th omit
- $\underline{3\text{th}}$. $\dim(\sigma^\vee) = r$

Toric variety

$$\mathbb{C}^* = (\mathbb{C} - \{0\}, \times)$$

$N \cong \mathbb{Z}^r$, r-dim algebraic torus

$T_N \cong (\mathbb{C}^*)^{\times n}$

$$T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$$

$\forall t_1, t_2 \in T_N$, $\forall m \in M$

$$t_1 \cdot t_2 := t_1(m) \cdot t_2(m).$$

$\forall m_1, m_2 \in M$, $a, b \in \mathbb{Z}$

$$t(am_1 + bm_2) = t(m_1)^a\, t(m_2)^b$$

$\forall m \in M$, character $\mathbf{e}(m)$.

$$\mathbf{e}(m) : T_N \longrightarrow \mathbb{C}^*$$
$$t \longmapsto t(m), \ \forall t \in T_N.$$

$$\mathbf{e}(am_1 + bm_2)(t_1^c t_2^d)$$
$$= (\mathbf{e}(m_1)(t_1))^{ac} (\mathbf{e}(m_1)(t_2))^{ad}$$
$$(\mathbf{e}(m_2)(t_1))^{bc} (\mathbf{e}(m_2)(t_2))^{bd}.$$