Asymptotic Theory.

L, $|L^{\otimes m}|$ $m \to \infty$

$\phi : X \dashrightarrow Y$.

Def.

$N(L) = N(X, L) = \{m \geq 0 : H^0(X, L^{\otimes m}) \neq 0\}$.

$N(L) = \{0\}$. $\forall m > 0$. $\underline{\quad} \equiv 0$

$N(L) \neq \{0\}$. $e = e(L) \to$ exponent of L.

$\forall n \gg 0$ $e | n$.

$D \longleftrightarrow \mathcal{O}_X(D)$ $N(D) = N(X, D)$

eg: T proj var. $\dim = d$. $\mathcal{Y}$. $\mathcal{Y}^{\otimes e} \simeq \mathcal{O}_T$.

smallest

Y ——— (k) B very amp

$X = Y \times T$, $L = pr_1^*(B) \otimes pr_2^*(\mathcal{Y})$

$\otimes^n \simeq \mathcal{O}$

$e(L) = e$, $N(L) = N_e$.

smallest. $L^{\otimes n}$

□.

$m \in N(X, L)$.

$\phi_m = \phi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$.

$Y_m = \phi_m(X) \subseteq \mathbb{P}H^0(X, L^{\otimes m})$.

closure.

Def. (Iitaka dim)

X normal,

$\kappa(L) = \kappa(X, L) = \max_{m \in N(L)} \{\dim \phi_m(X)\}$

X normal,

$$\kappa(L) = \kappa(X, L) = \max_{m \in N(L)} \{\dim \phi_m(X)\}$$

$N(L) \neq \{0\}$. $\quad = (0)$. $\kappa(L) = -\infty$.

If X is non normal. $\nu : X' \longrightarrow X$.

$$\kappa(X, L) = \kappa(X', \nu^* L).$$

$\kappa(X, L) = -\infty$. $\quad 0 \leq \kappa(X, L) \leq \dim X$.

eg: $\kappa(X, L) = \dim(Y) = k$

K_X. Kodaira dim $\kappa(X, K_X)$.

eg: $X = Bl_p \mathbb{P}^2$. $\underline{H}$, E.

$$\mathcal{O}_X(H), \quad \underline{\mathcal{O}_X(H+E)} \longrightarrow \mathcal{O}_E(-1).$$

$\kappa = -\infty$.

$$\downarrow$$

$\underline{\mathcal{O}_X(H)|_E} \longrightarrow \dim = 0$.

$0 \to \mathcal{I}_E \to \mathcal{O}_X \to \mathcal{O}_E \to 0$

$\quad \mathcal{I}_E = \mathcal{O}(-E)$

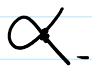
$0 \to \mathcal{O}_X(H-E) \to \mathcal{O}_X(H) \to \underline{\mathcal{O}_E(H.E)} \to 0$

$\quad \mathcal{O}_E(H.E) = \mathcal{O}_E$ $\longrightarrow$

$X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = pr_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(1)$

$\kappa(X, L) = -\infty$.

$Y = \mathbb{P}(\,) \times \mathbb{P}^1$. $\quad \kappa(X, L|_Y) = 1$.

eg: 

X nodal cubic curve.

$\nu : X' \longrightarrow X$.

$\quad \mathbb{P}^1$

$\nu' : X' \to X.$
$\quad \mathbb{P}^1$

$\qquad\qquad\qquad\qquad m>0$

$\underline{L} \in \mathrm{Pic}^0(X) \cong \mathbb{G}_m . \qquad H^0(X, L^{\otimes m}) = 0.$

non-torsion. lbd deg 0.

$L' = \nu^* L = \mathcal{O}_{\mathbb{P}^1} . \quad \to \kappa(X', L') = 0.$

Def:. Algebraic fibre space.

$f : X \to Y$ surj. proj. $\quad \underline{f_* \mathcal{O}_X = \mathcal{O}_Y}.$

red. irr.

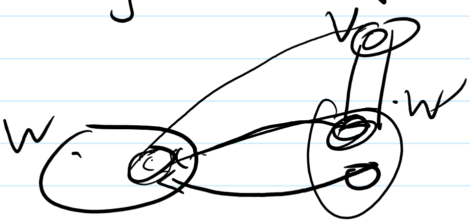
$\forall y \in Y, \quad \underline{f^{-1}(y)}$ connected.

$$\begin{tikzcd} V \arrow[r, "\forall g"] \arrow[d, "a"'] & W \\ W' \arrow[ur, "b"'] & \end{tikzcd}$$

Stein factorization. Zariski Main Thm.

alg. fibre space $\xleftarrow{} a$; $b \to$ finite.

g self is alg fibre space iff b trivial.



Y normal. $\forall$ proj surj $f : X \to Y$ connected fibre. is a fibre space

If $f : X \to Y$. alg fibre space. with normal, & if $\mu : X' \to X$. birat.
$\Rightarrow f \circ \mu : X' \to Y$ alg fibre space

Lemma. (Pull back via a fibre space).
$f : X \to Y$, alg fibre space. , L lbd on Y. Then
$H^0(X, f^*L^{\otimes m}) = H^0(Y, L^{\otimes m}) . \ \forall m \geq 0.$

alg fibre space.

$$H^0(X, f^*L^{\otimes m}) = H^0(Y, L^{\otimes m}) \quad \forall m \geq 0.$$

$\kappa(Y, L) = \kappa(X, f^*L)$.

pf: $H^0(X, f^*L^{\otimes m}) = H^0(Y, f_*(f^*L^{\otimes m}))$

$$f_*(f^*L^{\otimes m}) = f_*\mathcal{O}_X \otimes L^{\otimes m}. \leftarrow \text{proj formula.}$$

& $f_*\mathcal{O}_X = \mathcal{O}_Y$.

$\square$

eg: (Inj of Pic Grp).

X, Y. irr. proj var $f: X \to Y$ alg fibre space.

$$f^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X) \quad \text{inj.}$$

B on Y lbd. $\quad f^*B \cong \mathcal{O}_X$. $H^0(Y, B) = H^0(X, f^*B) \neq 0$.

$$H^0(Y, B^{-1}) = H^0(X, f^*B^{-1}) \neq 0.$$

$$B = \mathcal{O}_Y.$$

$\square$

Def. Section Ring ~ line.

L. proj var X.

$$R(L) = R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$$

D. sim for CDiv.

eg: $\mathbb{P}^n = X$. $L = \mathcal{O}_{\mathbb{P}^n}(1)$, $R(L) = \mathbb{C}[T_0, \ldots, T_n]$

homogeneous coordinate ring of $\mathbb{P}^n$.

Def (F.g. lbd & div). $\quad R(\mathcal{O}_X(D))$ $\quad$ Canonical ring. Bundle.

L, (D). $\longrightarrow$ $R(L) / R(D)$. f.g. $\quad$ Birk. 2010

Def: (Stable base locus) of D.

alg set.

$$\mathbb{B}(D) = \bigcap_{m \geq 1} \underline{Bs(|mD|)}.$$

set-theory.

Prop. $\mathbb{B}(D)$ is unique minimal element of

$$\underline{\{Bs(|mD|)\}_{m \geq 1}}.$$

$\exists\, m_0$ s.t.

$$\mathbb{B}(D) = Bs(|km_0 D|) \quad \forall k \geq 1.$$

pf: $\forall m, l \geq 1$,

$$Bs(|lmD|) \subseteq Bs(|mD|).$$

reverse

$$\mathfrak{b}(|mD|)^l \subseteq \mathfrak{b}(|mlD|).$$

if $Bs(|pD|)$ & $Bs(|qD|)$ both minimal.

$$\begin{array}{ccc} Bs(|pD|) & & Bs(|qD|) \\ \| & & \| \\ & Bs(|pqD|) & \end{array}$$

□

eg: D. conv. $\kappa(X, D) \geq 0$.

$$\mathbb{B}(pD) = \mathbb{B}(D). \quad \underline{\forall p} \geq 1.$$

□

eg. (scheme structure)

$Bs(|mD|) \longleftrightarrow \mathfrak{b}(|mD|) \subseteq \mathcal{O}_X$

$X = Bl_p \mathbb{P}^2$. E, H. $\quad D = E + H$.

$$\mathfrak{b}(|mD|) = \mathcal{O}_X(-mD)$$

$\underline{m\text{-th}.\text{-order}}\cdot mE$ of E.

□

July 1st. Ann [illegible]