Infinitesimal Okounkov bodies

X irr proj var $\dim = d$ $x \in X$ $V_\bullet$

$$T_x X = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{d-1} \supseteq \{0\}$$

Blowup. $\mu : X' = Bl_x(X) \longrightarrow X$, E.

$F_\bullet = F(x, V_\bullet)$ in X' $\mu^* D \longleftarrow D$

$F_\bullet$: $\boxed{X' \supseteq E = \mathbb{P}_{sub}(T_x X) \supseteq \mathbb{P}_{sub}(V_1) \supseteq \cdots \supseteq \mathbb{P}_{sub}(V_{d-1}) = \{pt\}}$

☆

$D \in \mathrm{Div}(X)$. $D' = \mu^* D$.

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X', \mathcal{O}_{X'}(mD')), \ \forall m.$$

flag $Y_\bullet'$ on X':

$$\Delta_{Y_\bullet'}(D) \overset{\text{def}}{=} \Delta_{Y_\bullet'}(D')$$

$F_\bullet$ $\Delta_{F_\bullet}(D) \subseteq \mathbb{R}^d$.

$$\Delta_{F_\bullet}(X) \subseteq \mathbb{R}^d \times N^1(X)_{\mathbb{R}}.$$

Prop. $D \in$ big $\mathrm{Div}(X)$.

$\Delta_{F(x, V_\bullet)}(D) \subseteq \mathbb{R}^d$ coincide

$\forall$ general choice $x \in X$ & flag $V_\bullet$,

holds for $\Delta_{(F, V_\bullet)}(X)$.

Def. $\Delta'(D) \subseteq \mathbb{R}^d$, $\Delta'(X) \subseteq \mathbb{R}^d \times N^1(X)_{\mathbb{R}}$.

$\Delta_{F(x, V_\bullet)}(D)$. $\Delta_{F(x, V_\bullet)}(X)$ for very general choice of $F(x, V_\bullet)$.

pf: by Thm. last discussion, OK6.

hold for countable big div D on X.

□

eg: $X = \mathbb{P}^3$, $x_0 : x_1 : x_2 : x_3$

$x = [1:0:0:0]$ $I_x = (x_1, x_2, x_3)$.

$x = [1:0:0:0]$ $\qquad I_x = (x_1, x_2, x_3)$.

$T_x X = (m_x/m_x^2)^\vee = \langle x_1^*, x_2^*, x_3^* \rangle$

F: $\underbrace{T_x X}_{V_0} \supseteq \underbrace{\langle x_1^*, x_2^* \rangle}_{V_1} \supseteq \underbrace{\langle x_1^* \rangle}_{V_2} \supseteq \underbrace{\{0\}}_{V_3}$

$Bl_x X = \mathrm{Proj}(k[x_0:x_1:x_2:x_3] \otimes k[y_0:y_1:y_2] / (x_1 y_0 - x_2 y_1, x_2 y_1 - x_3 y_2))$

$E = \underline{(1:0:0:0) \times (y_0:y_1:y_2)}$.

$P_{sub}(E) = (1:0:0:0) \times (y_0, y_1, y_2)$

$P_{sub}(V_1) = (1:0:0:0) \times (y_0:y_1:0)$ $\qquad X. \quad E$

$P_{sub}(V_2) = (1:0:0:0) \times (1:0:0)$

Conditions on Okounkov bodies on surfaces.

Zariski decomposition

S smooth surface. (C, x). smooth C. $x \in C$.

D pseudo-eff real div on S.

$$D = \underbrace{P(D)}_{\text{nef}} + \underbrace{N(D)}_{0 \text{ or eff}}.$$

0 or eff $\sim$ neg-def matrix

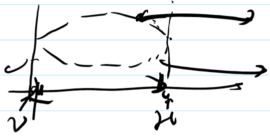
$(P(D).E) = 0$ for $\forall E$ component in $N(D)$

Minimality: $D = \underbrace{M}_{\text{nef}} + \underbrace{N}_{\text{eff}} \Rightarrow N - N(D)$ is eff.

ν we of C in $N(D)$

$\mu = \mu(D; C) = \sup\{t > 0 : D - tC \text{ is big}\}$.

$\mu(D)$.



$\beta(t) = P_t C + \mathrm{ord}_x(N_t|_C)$

$\alpha(t) = \mathrm{ord}_x(N_t|_C)$

$\forall t \in [\nu, \mu]$. $\qquad D_t = D - tC$

Zar dec

$= P_t + N_t$.

$\Delta_{(C,x)}(S; D) \subseteq \mathbb{R}^2$

$= \{(t, y) \in \mathbb{R}^2 : \nu \le t \le \mu, \ \alpha(t) \le y \le \beta(t)\}$.

$\Delta_{(C,x)}(S;D) \subseteq \mathbb{R}$
$= \{(t,y) \in \mathbb{R}^2 : \nu \le t \le \mathcal{H},\ \alpha(t) \le y \le \beta(t)\}.$

$D' = D - \mathcal{H}C.$

$\forall \underline{t} \in [\nu, \mathcal{H}].\quad s = \mathcal{H} - t.$

$$D_s' \overset{\text{def.}}{=} D' + sC = D' + (\mathcal{H}-t)C = D - tC$$

The segmen $\{D_t : t \in [\nu, \mathcal{H}]\}.$ in $\{D_s' : s \in [0, \mathcal{H}-\nu]\}$

$D_s' = P_s' + N_s'$ Zar Decomp

Prop. $s \mapsto N_s'$ $\underbrace{\text{"decrea"}}_{\downarrow}$ on $[0, \mathcal{H}-\nu]$

$0 \le s' < s \le \mathcal{H}-\nu.\quad N_{s'}' - N_s'$ eff.

$n.\ \#\underbrace{\{N_0'\}}_{\text{irr. comp.}}$, $(p_i)_{0 \le i \le \underbrace{k}}$ of $[0, \mathcal{H}-\nu]$ $\quad k \le n.$

$A_i,\ \underbrace{B_i}_{\substack{\downarrow \\ \text{rational}}}$ Div.

$\underline{N_s' = A_i + sB_i}.\quad s \in [p_i, p_{i+1}].$

pf: $\underline{C_1, \dots, C_n}$. irr comp of $\mathrm{Supp}(N_0')$.

s', s s.t. $0 \le s' < s \le \mathcal{H}-\nu.$

$$\underbrace{P_{s'}'}_{\substack{\downarrow \\ \text{nef.}}} = D_{s'}' - N_{s'}' = (D_s' - (s-s')C) - N_{s'}'$$
$$= D_s' - \underbrace{\{(s-s')C + N_{s'}'\}}_{\substack{\downarrow \\ \text{effective}}}$$

neg part minimal.

To show C not in $\mathrm{supp}(N_s')$. for $s \in [0, \mathcal{H}-\nu]$.

If $\underline{C \text{ in}}$, for some s. $N'_{\mathcal{H}-\nu}$

$\forall \lambda > 0.$

$$D_{s+\lambda}' \overset{\text{Zar dec}}{=} P_s' + (N_s' + \lambda C)$$

contradict to def of ν.

Rearrange. $\{C_i\}$ $\quad \mathrm{supp}\{N_{\mathcal{H}-\nu}'\}$. consists of $C_{k+1}, \dots, C_n$.

$$p_i \overset{\text{def}}{=} \sup\{s : C_i \le \mathrm{Supp}(N_s')\}.\ \forall i = 1, \dots, k.$$

WLOG. $0 = p_0 < p_1 \le \dots \le p_{k-1} \le p_k \le \mathcal{H}-\nu.$

WLOG. $0 = p_0 < p_1 \leq \cdots \leq p_{k-1} \leq p_k \leq H - \nu$.

To: N_s' is linear on $[p_i, p_{i+1}]$.

$\hookrightarrow$ on $(\quad , \quad)$

If $s \in \longrightarrow$. $\mathrm{Supp}(N_s') \subseteq \{C_{r+1}, \ldots, C_n\}$.

$N_s' \xleftarrow{\text{determ } \exists !}$ $N_s' \cdot C_j = (D' + sC) \cdot C_j$, $\forall r+1 \leq j \leq n$.

$I(C_{r+1}, \ldots, C_n)$ non-degenerate.

$\exists !$ A_v, B_v supp on $\bigcup_{r+1}^{n} C_j$. s.t.

$A_i \cdot C_j = D' \cdot C_j$ & $\underset{\downarrow}{B_i} \cdot C_j = C \cdot C_j$. for all $r+1 \leq j \leq n$.

rational. $N_s' = A_v + s B_v$. $\forall s \in (p_v, p_{v+1})$ $\square$

PHI10. α, β.

Zariski cham

Find.

Eg. $\mathbb{P}^2$, blowup 2 pts. big cone 3-dim space

amp divs.

$mD_1 - nD_2$.

E. –

big $\Rightarrow$ positive vol.