OK6

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Fujita Approximation

L lbd on V, $n = \dim V$

$$h^0(V, tL) \geq \frac{d t^n}{n!} + \phi(t) \quad ; \quad \lim_{t \to \infty} \frac{\phi(t)}{t^n} \to 0.$$

$\forall \varepsilon > 0$, birm. $\pi : M \to V$. eff div E on M.

s.t. $H = \pi^* L - E$: semi-ample $\mathbb{Q}$-div $H^n > d - \varepsilon$.

small

rmk: If $H^0(t\pi^* L) \overset{(\leq)}{=} H^0(tH)$ $\forall t > 0$, s.t. tE, $\mathbb{Z}$ div

$\pi^* L = E + H$ Zar decr of L.

idea: $tH \subseteq t\pi^* L$

$+E$ $\to$ fixed compon. $\leftarrow$ difficult in high dim.

$\Gamma \subset \mathbb{Z}^{d+1}_{\geq 0}$ $\Delta = \Delta(\Gamma)$, and $\Gamma_m \subseteq \mathbb{Z}^d_{\geq 0}$.

Prop Γ satisfies Lemm 2.2. (D)

1. $\Gamma_0 = \{0\}$
2. $\{(v_i, 1)\}$ finite generate B, $\Gamma \subseteq B$,
3. Γ group $\mathbb{Z}^{d+1}$.

$\forall$ fix $\varepsilon > 0$, $\exists\, p_0 = p_0(\varepsilon)$, $\forall p \geq p_0$

$$\lim_{k \to \infty} \frac{\#(k * \Gamma_p)}{k^d p^d} \geq \mathrm{vol}_{\mathbb{R}^d}(\Delta) - \varepsilon.$$

" $k * \Gamma_p$, $\{\alpha\} \in \Gamma_p$,

$\sum_{i=1}^{k} \alpha_i$

Lemma. If $\Gamma \subseteq \mathbb{Z}^{d+1}_{\geq 0}$ gen $\mathbb{Z}^{d+1}$.

$$\Gamma_m \subseteq \mathbb{Z}^{d+1}_{\geq 0} \to \mathbb{Z}^d \quad \text{for } m \gg 0$$

pf. Γ f.g. $\left(\bigcup \Gamma_m \to P. \right)$

$$\Sigma = \Sigma(\Gamma) \subseteq \mathbb{R}^{d+1}$$

$$(\Sigma + y) \cap \mathbb{Z}^{d+1}_{\geq 0} \subseteq \Gamma.$$

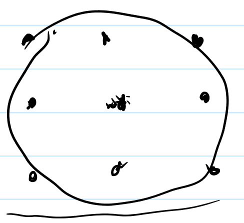
$y \in \Gamma$.

Δ:

$$(\Sigma + y)_m = (\Sigma + y) \cap (\mathbb{R}^d \times \{m\}) \subseteq \mathbb{R}^d.$$

$\cup|$

$B(r)$ $\quad r > 2\sqrt{d}$. $\quad \mathbb{Z}^d$.



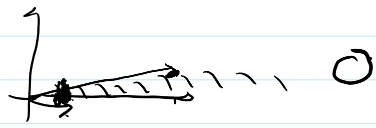
dim = 1.

dim 2

(1, 0)

(n, 1)

$\Delta \neq \emptyset$.



NEWTON-OKOUNKOV BODIES, SEMIGROUPS OF INTEGRAL POINTS, GRADED ALGEBRAS AND INTERSECTION THEORY-KIUMARS KAVEH, A. G. KHOVANSKII

pf: Γ f.g. Given p, let

$$\Theta_p = \mathrm{conv}(\Gamma_p) \subseteq \mathbb{R}^d.$$

$$\lim_{p \to \infty} \frac{\mathrm{vol}_{\mathbb{R}^d}(\Theta_p)}{p^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta)$$

$\underset{\sim}{\Gamma_p} \xrightarrow[\text{grp}]{\text{gen}} \mathbb{Z}^d$. $\underline{p} \gg 0$.

$$\lim_{k\to\infty} \frac{\#(k * \Gamma_p)}{k^d} = \mathrm{vol}_{\mathbb{R}^d}(\Theta_p).$$

$\forall \varepsilon > 0$, $\exists\ p_0 = p_0(\varepsilon)$ s.t.

$$\lim_{k\to\infty} \frac{\#(k * \Gamma_p)}{p^d k^d} \geq \mathrm{vol}_{\mathbb{R}^d}(\Delta) - \frac{\varepsilon}{2},$$

$\underline{\forall p} > p_0$

$$\mathrm{vol}(\Delta') \geq \mathrm{vol}(\Delta) - \frac{\varepsilon}{2}$$

$\Box$

Thm. D big. irr proj var X $\dim = d$.

$p, k > 0$.

$$V_{k,p} = \mathrm{Im}\left(S^k H^0(X, \mathcal{O}_X(pD)) \longrightarrow H^0(X, \mathcal{O}_X(pkD))\right)$$

Given $\varepsilon > 0$; $\exists p_0 = p_0(\varepsilon)$ s.t. $\forall p \geq p_0$

$$\Rightarrow \lim_{k\to\infty} \frac{\dim V_{k,p}}{p^d k^d / d!} \geq \mathrm{vol}_X(D) - \varepsilon.$$

pf: $Y_\bullet$ on X, $\Gamma = \Gamma(D)$ $\sim$ $\nu_{Y_\bullet} = \nu$.

$\Gamma_p \sim \mathcal{O}_X(pD)$

$\hookrightarrow = \mathrm{Im}\left((H^0(X, \mathcal{O}_X(pD)) - \{0\}) \xrightarrow{\nu} \mathbb{Z}^d_{\geq 0}\right).$

Given $s_1, \dots, s_k \in H^0(X, \mathcal{O}_X(pD))$

$$\nu(s_1 \cdots s_k) = \sum \nu(s_i),$$

follows that

$$k * \Gamma_p \subseteq \mathrm{Im}\left((V_{k,p} - \{0\}) \xrightarrow{\nu} \mathbb{Z}^d_{\geq 0}\right).$$

$(\underline{\dim W} \sim \#$ val vectors.$)$

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta) = \mathrm{vol}_X(D)/d!.$$

by prop. $\square$.

Thm. X irr var dim d. $W_\bullet$ graded linear series $\sim D$ on X.

$$V_{k,p} = \mathrm{Im}(S^k W_p \to W_{kp})$$

$W_\bullet$ ..., fix $\varepsilon > 0$. $\exists\, p_0 = p_0(\varepsilon)$. s.t.

$\forall p \geq p_0$, then

$$\lim_{k\to\infty} \frac{\dim V_{k,p}}{p^d k^d / d!} \geq \mathrm{vol}_X(W_\bullet) - \varepsilon. \qquad \square.$$

For multiplicities of graded families of ideals

X irr var of dim d.

graded family of ideals. $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}$ on X.

$$\mathfrak{a}_k \subseteq \mathcal{O}_X. \quad \mathfrak{a}_0 = \mathcal{O}_X.$$

$$\mathfrak{a}_k \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{k+l}. \qquad \forall\, k, l \geq 0.$$

eg. $\mathfrak{a}_k = \{ f \in \mathcal{O}_X : \nu(f) \geq k \}$.

fixed $x \in X$. $\sim \mathfrak{m}$., $\mathfrak{a}_\bullet$.

$\mathfrak{a}_k$ is $\mathfrak{m}$-primary.

Then $\mathfrak{a}_k \subseteq \mathcal{O}_X$ finite codim.

$$\mathrm{mult}(\mathfrak{a}_\bullet) = \limsup_{m\to\infty} \frac{\dim_k(\mathcal{O}_X/\mathfrak{a}_m)}{m^d/d!}.$$

Thm. $\mathrm{mult}(\mathfrak{a}_\bullet) = \lim_{p\to\infty} \dfrac{e(\mathfrak{a}_p)}{p^d}$?

Lemma X proj var. $\mathfrak{a}_\bullet$ m-primary

$\exists$ amp div D on X. s.t. $\forall p, k > 0$.

$$\underline{H^i(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_p^k)} = 0, \text{ for } i > 0.$$

Moreover. rat

$$\phi_p : X \dashrightarrow \mathbb{P} = \mathbb{P}H^0(X, \mathcal{O}_X(pD))$$

defined by $H^0(X, \mathcal{O}_X(pD) \otimes \mathfrak{a}_p) \subseteq H^0(X, \mathcal{O}_X(pD))$

is birat over image

pf:

$\mathfrak{a}_1^{kp} \subseteq \mathfrak{a}_p^k$. by def.

Since $\underline{\mathfrak{a}_p^k / \mathfrak{a}_1^{kp}}$ 0-dim. support:

$$H^i(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_1^{pk}) \longrightarrow H^i(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_p^k)$$

is surjective. $i > 0$.

$$0 \to \cdots \otimes \mathfrak{a}_1^{pk} \to \cdots \otimes \mathfrak{a}_p^k \to \cdots \underbrace{\otimes\, \mathfrak{a}_p^k / \mathfrak{a}_1^{pk}}_{\dim = 0} \to 0$$

$$\to H^i \longrightarrow H^i \longrightarrow H^i \xrightarrow{i>0}$$

$$\underline{H^i} = 0 \quad \forall i > 0$$

$p = 1$.

$$\mu : X' = Bl_{\mathfrak{a}_1}(X) \longrightarrow X.$$

$$E \subseteq X'.$$

D_0 amp div on X.

$-E$. amp for μ.

$\mu^* mD_0 - E$. is ample. div X'. $\forall m \geq 1$.

$$\mu_* (\mathcal{O}_{X'}(-kE)) = \mathfrak{a}_1^k.$$

$$R^i \mu_* (\mathcal{O}_{X'}(-kE)) = 0 \quad (i>0).$$

$k \gg 0$.

Fujita vanishing on X.

Leray spectral seq.

$$\Rightarrow H^i(X, \mathcal{O}_X(kmD_0) \otimes \mathfrak{a}_1^k) = 0. \quad (i>0).$$

$\forall\ m \geq 1$, $k \gg 0$.

By $m \geq m_1$ for $m_1 > 0$. $\forall k$.

$$D = mD_0 \qquad m \geq m_1.$$

$$H^0(X, \mathcal{O}_X(pD)) \otimes \mathfrak{a}_1^p) \subseteq H^0(X, \mathcal{O}_X(pD)) \otimes \mathfrak{a}_p).$$

$p=1$. $\Rightarrow$ biran ϕ_p.

$\square$.

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pf: Affine neighbourhood of x. $\Rightarrow X$ proj.

D amp. divisor, set

$$W_m = H^0(X, \mathcal{O}_X(mD) \otimes \mathfrak{a}_m).$$

$\llcorner$ above lemma

Conv(B).

$$V_{k,p} = \mathrm{Im}\,(S^k(H^0(X, \mathcal{O}_X(pD) \otimes \mathfrak{a}_p)) \\ \longrightarrow H^0(X, \mathcal{O}_X(pkD) \otimes \mathfrak{a}_{pk})).$$

factor through $H^0(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_p^k)$

$$\Rightarrow \quad V_{k,p} \simeq H^0(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_p^k).$$

By $H^1(\cdots \overline{kpD} \otimes \mathfrak{a}_p^k) = 0.$

$$\dim V_{k,p} \leq h^0(X, \mathcal{O}_X(kpD) \otimes \mathfrak{a}_p^k)$$
$$= h^0(\cdots(kpD)) - \dim(\mathcal{O}_X/\mathfrak{a}_p^k)$$

$$\Rightarrow \lim_{k\to\infty} \frac{\dim V_{k,p}}{p^d k^d/d!} \leq \mathrm{vol}_X(D) - \frac{e(\mathfrak{a}_p)}{p^d}.$$

$$\mathrm{vol}(W_\bullet) = \mathrm{vol}_X(D) - \mathrm{mult}(\mathfrak{a}_\bullet).$$

$\Rightarrow$ $\varepsilon > 0$, $\exists\, p_0 = p_0(\varepsilon)$

s.t. $\dfrac{e(\mathfrak{a}_p)}{p^d} \leq \mathrm{mult}(\mathfrak{a}_\bullet) + \varepsilon. \quad p \geq p_0$

$\quad\geq$

$\mathrm{mult}(\mathfrak{a}_\bullet).$

$\square$.

Generic Infinitesimal Flags.

$\pi : X \longrightarrow T$. flat. surj. mor of var.
rel dim $\underline{d}$.

$\mathcal{D}$ be CDiv on X, flat over T.

$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_d.$

Y_i: codim $= i$ in X. flat surj over T.

$t \in T$.

$t \in T$,

$$X_t = \pi^{-1}(t), \quad D_t = D|_{X_t}.$$

$$Y_{\nu,t} = \pi^{-1}(t) \cap Y_\nu$$

Assume T irr, $\forall \nu \in T$.

(i). X_t, $Y_{\nu,t}$ are red, irr.

(ii). Each $Y_{\bullet,t}$ admissible flag on X_t,

(iii). $\forall \nu$, $Y_{\nu+1}$ is $\mathrm{CDiv}(Y_\nu)$.

$(Y_{\nu+1,t}, Y_t)$

$$\Delta_{Y_{\bullet,t}}(X_t; D_t) \subseteq \mathbb{R}^d.$$

Thm. π is projective. D_t big on X_t $\forall t \in T$. $\exists$ $\mathcal{B} = \bigcup B_m \subset T$.

countable union, proper Zar closed. $B_m \subsetneq T$

$\Delta_{Y_\bullet,t}(X_t; D_t)$ all coincide for $t \notin \mathcal{B}$, i.e.

$\subseteq \mathbb{R}^d$. independent of t, $t \in T - B$.

Lemma.

E be $\mathrm{CDiv}(X)$ flat T. fix $\sigma \in \mathbb{Z}^d$.

$\emptyset \neq U \subseteq T$. s.t.

$$\dim H^0(X_t, \mathcal{O}_X(E_t))^{\geq \sigma}. \quad E \sim t.$$

are constant for $t \in U$. $\forall f$, $\nu(f) \geq \sigma$.

pf. $\mathcal{L} = \mathcal{O}_X(E)$. $L_t = \mathcal{L}|X_t$.

$\forall,\ \mathcal{L} = \mathcal{O}_X(\mathcal{E}),\ \mathcal{L}_t = \mathcal{L}|X_t.$

$Y_\bullet$ partial flag on X. $\mathcal{L}^{\geq 0} \subseteq \mathcal{L}$.

flat over T.

$$\mathcal{L}^{\geq 0} \otimes \mathcal{O}_{X_t} = (\mathcal{L}_t)^{\geq 0}, \quad \forall t \in T.$$

Since.

$$H^0(X_t, \mathcal{L}_t^{\geq 0}) = H^0(X_t, \mathcal{L}_t)^{\geq 0},$$

(semicontinuity thm).

$\square$

pf: $m \geq 0$

$$\underline{\nu_{Y_\bullet, t}} : (H^0(X_t, \mathcal{O}_{X_t}(mD_t)) - \{0\}) \longrightarrow \mathbb{Z}^d.$$

$\exists\, U_m:\ \forall t\ \ \nu_{Y_t}$ independent t.

$$B_m = T - U_m$$

$\exists\, U_m' \subseteq T.\quad \dim H^0(X_t, \mathcal{O}_X(mD_t))$ constant

$\mathrm{im}(\nu_{Y_\bullet, t})\ \forall t \in U_m'$

$\underline{\text{lem}}:\ \underline{\rightarrow \text{bounded in } \mathbb{R}^d} \Rightarrow \underline{\text{fixed finite}}$ set in $\mathbb{Z}^d$.

$\square$

Convex bodies appearing as Okounkov bodies of divisors alex k uronya, victor lozovanu, and catriona maclean

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