OK5

2024年4月16日 16:48

4.4. Slices

irr. red. CDiv.

X irr proj var $\dim = d$., $E \subseteq X$

$$Y_\bullet : X \supseteq \underset{\substack{\| \\ Y_1}}{E} \supseteq Y_2 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{pt\}.$$

$\xi \in N^1(X)_{\mathbb{R}}$. big class.

$$\Delta(\xi) = \Delta(X)_\xi \subseteq \mathbb{R}^d.$$

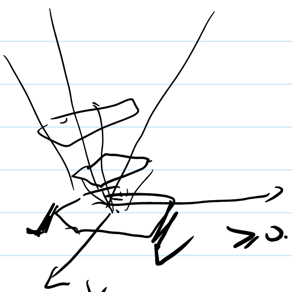
$pr_1 : \Delta(\xi) \longrightarrow \mathbb{R}$. $\quad (x_1, \ldots, x_n) \in \Delta(\xi) \mapsto x_1$

- $\Delta(\xi)_{\nu_1 = t} = pr_1^{-1}(t) \subseteq \{t\} \times \mathbb{R}^{d-1} = \mathbb{R}^{d-1}$
- $\Delta(\xi)_{\nu_1 \geq t} = pr_1^{-1}([t, +\infty)) \subseteq \mathbb{R}^d$.

$\xi - te$. $\in N^1(X)_{\mathbb{R}}$; e class E ; Y_1

Assume. $E \not\subseteq B_+(\xi) \Rightarrow \Delta(\xi)_{\nu_1 = 0} \neq \emptyset$.

$$\mu(\xi; e) = \sup\{s > 0 \mid \xi - se \in \mathrm{Big}(X)\}.$$



Thm. $E \not\subseteq B_+(\xi)$, $0 \leq t < \mu(\xi; e)$

$$\Rightarrow \boxed{\Delta(\xi)_{\nu_1 \geq t} = \Delta(\xi - te) + t \cdot \vec{e}_1}.$$

$\vec{e}_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^d_{\geq 0}$

$\Rightarrow$. $\Delta(\xi)_{\nu_1 = t} = \Delta_{X|E}(\xi - te)$. ✓ $\quad$ lbd. $|_E$

1E.

$\Rightarrow$. $\Delta(\xi)_{\nu_1=t} = \Delta_{X|E}(\xi - te)$. ✓

$\mathbb{P}^2$. $(\nu_1, (\nu_2, \nu_3)) = (1; 0; 0)$. $\mathcal{L} = \mathcal{O}(1)$

eg. ★.

ample ok..

Base locus.

Cor. (i). $\mathrm{vol}_{\mathbb{R}^{d-1}}(\Delta(\xi)_{\nu_1=t}) = \frac{1}{(d-1)!} \cdot \mathrm{vol}_{X|E}(\xi - te)$.

(ii). For any $0 < a < \mu(\xi; e)$.

$$\mathrm{vol}_X(\xi) - \mathrm{vol}_X(\xi - ae) = d \cdot \int_a^0 \mathrm{vol}_{X|E}(\xi + te)\, dt.$$

(iii). $t \mapsto \mathrm{vol}_X(\xi + te)$. differenciable at $t=0$

$$\frac{d}{dt}(\mathrm{vol}_X(\xi + te))\Big|_{t=0} = d \cdot \mathrm{vol}_{X|E}(\xi)$$

pf:

(i) $\mathrm{vol}_{\mathbb{R}^{d-1}}(\Delta_{X|E}(\xi)) = \frac{1}{d!} \mathrm{vol}_{X|E}(\Delta_{X|E}(\xi))$ $\Big\} \Rightarrow$ (i)

Thm.

(ii) $(d-1)$-dim vol. of fibres orthogonal proj to $\mathbb{R}$.

(iii). $E \Rightarrow E \notin B_+(\xi + \varepsilon e)$. $0 < \varepsilon \ll 1$.

$\xi \to \xi + \varepsilon e$. □.

$\Gamma \subseteq \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $a > 0$. $\begin{cases} \Gamma_{\nu_1 \geq a} \subseteq \Gamma .. \\ \Gamma_{\nu_1 = a} \subseteq \Gamma. \end{cases}$

$$\Gamma_{\nu_1 \geq a} = \{(\nu_1, \dots, \nu_d, m) \in \Gamma \mid \nu_1 \geq am\}$$

$$\Gamma_{\nu_1 = a} = \{ \cdots \quad \text{———} \quad \mid \nu_1 = am\}.$$

pf: Show $t > 0$ $\xi \to \xi + \varepsilon e$. $0 < \varepsilon \ll 1$ $t = 0$.

$\nu = \nu_{Y_\bullet}$, $Y_\bullet$

D. $a \in \mathbb{Z}_{\geq 0}$. s.t. $D - aE$ is big.

$D.\ a\in\mathbb{Z}_{\geq 0}$ s.t. $D-aE$ is big.

$\Rightarrow \forall\, m\geq 0.\quad H^0(X, \mathcal{O}_X(mD-maE)) \subseteq H^0(X, \mathcal{O}_X(mD))$

$= \{ s\in H^0(X, \mathcal{O}_X(mD)) \mid \mathrm{ord}_E(s)\geq ma \}$

$= \{ \quad \mid \nu_1(s)\geq ma \}$

$Y_\cdot$. $\Gamma(D)_{\nu_1\geq a}$ is image.

$\varphi_a : \mathbb{Z}^d_{\geq 0}\times\mathbb{Z}_{\geq 0}\longrightarrow \mathbb{Z}^d_{\geq 0}\times\mathbb{Z}_{\geq 0}$

$(\nu, m)\longmapsto (\nu+ma\vec{e}_1, m)$

$\varphi_a(\Gamma(D-aE))$.

$\mathbb{R}$

To cone

$$\Sigma(\Gamma(D)_{\nu_1\geq a}) = \varphi_{a,\mathbb{R}}(\Sigma(\Gamma(D-aE))$$

$$\Rightarrow \Delta(D-aE)+a\vec{e}_1 = \Delta(D)_{\nu_1\geq a}.$$

$$\Rightarrow \Delta(pD-qE)+q\vec{e}_1 = \Delta(pD)_{\nu_1\geq q}.$$

big.

(2) $D.\quad a>0$.

$\Gamma_{X|E}(D-aE)\subseteq \mathbb{Z}^{d-1}_{\geq 0}\times\mathbb{Z}_{\geq 0}$

w.r.t. $Y_\cdot|E$. $\quad\to \Delta_{X|E}(D-aE)$

$\nu_{Y_\cdot}$, $\Gamma(D)_{\nu_1=a}\subseteq \mathbb{Z}^d_{\geq 0}\times\mathbb{Z}_{\geq 0}$.

$\varphi(\cdot)$, $\mathbb{Z}^{d-1}_{\geq 0}\times\mathbb{Z}_{\geq 0}\longrightarrow \mathbb{Z}^d_{\geq 0}\times\mathbb{Z}_{\geq 0}$.

$\varphi : (\nu_2,\dots,\nu_d, m)\longmapsto (\underset{ma}{\underset{\uparrow}{\ }}, \nu_2,\dots, m)$

on X. sec on E. restrict on X.

$\star\quad \Sigma(\Gamma(D)_{\nu_1=a}) = \Sigma(\Gamma(D))_{\nu_1=a}$. $\quad\mathbb{R}^d\times\mathbb{R}$.

$\Delta(D)_{\nu_1=a} = \Delta_{X|E}(D-aE)$, $\clubsuit$.

$\Delta(pD)_{\nu_1=q} = \Delta_{X|E}(pD-qE)$., $pD-qE$ is big.

$\Delta(pD)_{\nu_1=q} = \Delta_{X|E}(pD - qE)$. , $pD - qE$ is big.
$q>0$.

Prop Appendix 1. ...

Eg: X smooth complex proj surface
D big $\mathbb{Q}$-div(X).
Zariski decom $D = P + N$.
P: not $\mathbb{Q}$-div. N: eff $\mathbb{Q}$-div.

mD, mN integral divisor. $\cdot h \longleftrightarrow mN$.

$$H^0(X, \mathcal{O}_X(mP)) \xrightarrow[\cong]{\cdot h} H^0(X, \mathcal{O}(D))$$

$C_i \in \mathrm{Big}(X)$. :
(i) $\forall i$. $T_1, \ldots, T_r$. s.t. $\forall$ $D \in \overline{C_i}$. negative part
of D support on $T_1 \cup \cdots \cup T_r$.

$D \to$ neg part of D linear or $\overline{C_i}$ with big cone.

$\boxed{\text{(ii). each } \overline{C_i} \text{ rational \& polyhedral} \quad \exists \text{ finitely such cones}}$

$X \supseteq C \supseteq \{x\}$.
C: irr. $\{x\}$: smooth on C.

(big)$^\circ$ $\quad \partial(\text{big})$ unknow.

Z. decom of D. O. body.

Lemma. D big $\mathbb{Q}$-div. on X. $D = P + N$.
$C \not\subseteq B_+(D)$ s.t. $C \not\subseteq \mathrm{Supp}(N)$.
$\alpha(D) = \mathrm{ord}_x(N|_C)$: ned

$$\alpha(D) = \mathrm{ord}_x(N|_C)$$

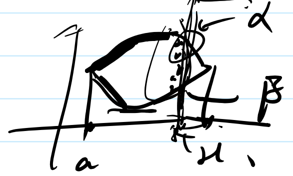
$$\beta(D) = \alpha(D) + (C\cdot P). \qquad \text{nef.}$$

Then. O. body

$$\Delta_{x|C}(D) = [\alpha(D), \beta(D)] \subseteq \mathbb{R}.$$

pf: [LM]: $\mathrm{vol}_{X|C}(D) = (C\cdot P)$.

β. $D - sC$ is big



cannot be clearer

$$\mu = \mu(D;C) = \sup\{s>0 \mid D - sC \text{ is big}\}$$

Thm. $\alpha, \beta : [a, \mu] \longrightarrow \mathbb{R}_+$.

$0 \le a \le \mu$, α convex, β concave.

$$\Delta(D) = \{(t,y)\in\mathbb{R}^2 : a \le t \le \mu,\ \alpha(t) \le y \le \beta(t)\}$$

α, β piecewise linear & rational $[a,\mu']$, $\mu' < \mu$.

$\Delta(D) \cap [a,\mu']\times\mathbb{R}$ is rational polytope.

pf: $t \in [0,\mu)$, $D_t = D - t\cdot C$.

$D_t = P_t + N_t$, Zar Decom.

a: ord of C in N_0. $\quad D - aC$ is big

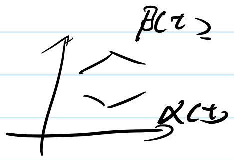
$$\Delta(D) = \Delta(D - aC) + (a, 0).$$

$D \to D - aC$. $\quad C \not\subset$ appear in N_0.

x ——— $N_t \quad t < \mu$.

Let. $\alpha(t) = \mathrm{ord}_x(N_t|_C)$

$\beta(t) = \alpha(t) + (C\cdot P_t)$.



$\Delta(D)$ region bounded by $\underset{\text{convex}}{\alpha(t)}$, $\underset{\text{concave}}{\beta(t)}$.

If $\alpha(\mu) = \min\{y \ge 0 : (\mu, y) \in \Delta(D)\}$,

$\beta(\mu) := \max\{y \ge 0 : (\mu, y) \in \Delta(D)\}$

α, β continuous on $[0, \mu]$.

α, β continuous on $[0, \mu]$. .

linearity

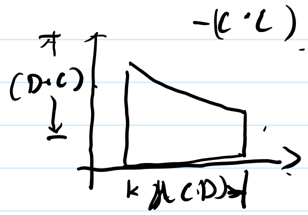
Patrycja Lusz

Patrycja Luszcz-Swidecka and David Schmitz: Minkowski decomposition of Okounkov bodies on surfaes
arXiv 1304.4246
Convex bodies appearing as Okounkov bodies of divisors alex k uronya, victor lozovanu, and catriona maclean
Arxiv 1008.4431, prop2.1
Prof Zariski decomposition.

Eg (abelian surfaces), D on abelian surface.

$D_t = D - tC$. nef. $\forall\ t \le \mu(D)$,

$-(C \cdot C)$ $\quad N \times$ occur.



amp no neg.

Cor. X smooth complex proj surface.

$X \supset \{C\} \supset [x]$. $\quad C \in N^1(X)$

class

If $\Delta(X) \subseteq N^1(X)_{\mathbb{R}} \times \mathbb{R}^2$. $\quad$ int $(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$

int $(\overline{\mathrm{Eff}}(X)) = \mathrm{Big}(X)$.

global Okbody of X.

$$\Delta(X) = \{(\xi, t, y) \mid \xi - tC \in \overline{\mathrm{Eff}}(X)\}$$

big

and $\Delta(X)$ rational polytope. neighborhood $\forall\ (\xi, t, y)$ ..

4.30.2024.