§4. Variation.

$\Delta(D)$. $\equiv_{num}$. of $D$.

4.1. $X$ irr proj var. $\dim = d$, $Y_\bullet$.

<u>Prop</u> $D \in \mathrm{bigDiv}(X)$.

(i). The NO $\Delta(D)$. depends only on. $\equiv_{num}$ class of $D$

(ii). For $\forall p \in \mathbb{Z}_{>0}$,

$$\Delta(pD) = p\Delta(D)$$

homothetic image of $\Delta(D)$.

rmk:

vol (i). $\mathrm{vol}(D) = \mathrm{vol}(D')$, if $D \equiv_{num} D'$

(ii). $\mathrm{vol}(aD) = a^n \mathrm{vol}(D)$.

$\mathrm{vol}(\Delta) \sim \frac{1}{n!}\mathrm{vol}(D)$

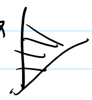
pf:

(i). $\Delta(D+P) = \Delta(D)$. $P$ num trivial div.

$\exists B$ s.t. $B + kP$ is very ample. $\forall k \in \mathbb{Z}$.

$a$. s.t. $aD - B \equiv_{lin} F$, eff. div..

$$(m+a)\cdot(D+P) \equiv_{lin} mD + (aD - B) + (B + (m+a)P).$$

$m \to \infty$ in ↗ 

$B + (m+a)P$ not pass $Y_i \subset Y_\bullet$.

$$\Gamma(D)_m + f \in \Gamma(D+P)_{m+a}.$$

$f$ valuation vector defining $F = aD - B$.

$Y_\bullet$

$\exists N$, $\forall$ num trivial $P$.

$H^0(X, \mathcal{O}_X(N+P)) \neq 0$.

$m \to \infty$

$\in \Delta(D) \subset \Delta(D+P)$.

$P \to D - P$.

$$p \to P - p.$$

$$\Delta(D) = \Delta(D+P).$$

(ii). [28. Lemma 22.38].

$r_0$ s.t. $|rD| \neq \emptyset$, $r > r_0$.

$q_0$ $\quad q_0 p - (p + r_0) > r_0$

$\Rightarrow$ $r \in [r_0+1, r_0+p]$.

$$E_r \in |rD|, \ F_r \in |(q_0 p - r)D|.$$

$\forall r \in [r_0+1, r_0+p]$.

$$|mpD| + \underset{\text{valuer}}{E_r} + E_r \subseteq |(mp+r)D| + E_r \subseteq |(m+q_0)pD|.$$

and

$$\Gamma(pD)_m + e_r + f_r \subseteq \Gamma(D)_{mp+r} + f_r \subseteq \Gamma(pD)_{m+q_0}$$

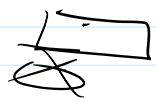
$m \to \infty$,

$$\Delta(pD) \subseteq p \cdot \Delta(D) \subseteq \Delta(pD).$$

$\square$

$\Delta(\xi) \subseteq \mathbb{R}^d$, $\quad \xi \in N^1(X)_{\mathbb{Q}}$: big class

Def. (Rational class).



$p >> 0$.

$$\Delta(\xi) = \frac{1}{p} \cdot \Delta(pD) \subseteq \mathbb{R}^d.$$

$\Delta(\cdot)$. $\quad D$. $\quad p$.

Prop. $\forall$ big class $\xi \in N^1(X)_{\mathbb{Q}}$,

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(\xi)) = \frac{1}{d!} \cdot \mathrm{vol}_X(\xi).$$

pf. $\quad D \xrightarrow{\text{rep}} \xi$, $\quad p >> 0$. $\quad pD$.

$$\begin{cases} \mathrm{vol}_X(\xi) = \frac{1}{p^d} \mathrm{vol}_X(pD). \\ \mathrm{vol}_{\mathbb{R}^d}(\Delta(\xi)) = \frac{1}{p^d} \mathrm{vol}_{\mathbb{R}^d}(\Delta(pD)). \end{cases}$$

$\square$.

§2. Global Okounkov body.

§2. Global Okounkov body.

$\Delta(\xi)$. $X$ irr proj var. $\dim = d$ $Y$.

Thm. Closed convex cone.

$$\Delta(X) \subseteq \mathbb{R}^d \times N^1(X)_{\mathbb{R}}$$

$$\begin{tikzcd} \Delta(X) \arrow[dr] & \subseteq & \mathbb{R}^d \times N^1(X)_{\mathbb{R}} \arrow[dl, "pr_2"] \\ & N^1(X) & \end{tikzcd}$$

fibre of $\Delta(X)$. $\xi \in N^1(X)_{\mathbb{Q}}$. $\Delta(\xi)$, i.e.

$$pr_2^{-1}(\xi) \cap \Delta(X) = \Delta(\xi) \subseteq \mathbb{R}^d \times \{\xi\} = \mathbb{R}^d.$$

Lemma. $X$ irr proj. dim $d$., $\overline{\mathrm{Eff}}(X)$. pointed.

i.e. if $0 \neq \xi \in \overline{\mathrm{Eff}}(X)$. $\Rightarrow -\xi \notin \overline{\mathrm{Eff}}(X)$

pf: If $d=1$.

$d=2$. eff cone = dual of nef cone.

$d \geq 3$, $\xi, -\xi \in \overline{\mathrm{Eff}}(X)$. $\Rightarrow (\xi \cdot C) = 0$, $\forall$ irr $C$ on $X$. ($C$: curve)

Show. $\exists$ irr $Y \subset X$, $C \subset Y$.

s.t. $\xi|_Y$. $-\xi|_Y \in \overline{\mathrm{Eff}}(Y)$.

$$\xi = \lim_{m\to\infty} d_m = -\lim_{m\to\infty} e_m.$$

$d_m \to D_m$ & $E_m \in \mathrm{eff\,Div}(X)_{\mathbb{R}}$.

$Y \supset C$. $Y \not\subset \mathrm{supp}(D_m)$ or $\mathrm{supp}(E_m)$.

$d \geq 3$. $K$. is uncountable. □

$N^1(X)_{\mathrm{fg}}$., $D_1, \cdots, D_r \in \mathrm{Div}(X)$. $\mathbb{Z}$-basis

$\mathbb{Z}_{\geq 0}$ - combination of $D_i$

$N^1(X) \cong \mathbb{Z}^r$ $N^1(X)_{\mathbb{R}} \cong \mathbb{R}^r$

$$N^1(X) \cong \mathbb{Z}^r \qquad N^1(X)_{\mathbb{R}} \cong \mathbb{R}^r.$$

$\overline{\mathrm{Eff}}(X)$. $\mathbb{R}^r_{\geq 0}$. $\qquad \vec{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$.

$$\vec{m} \cdot D = \sum_{i=1}^{r} m_i D_i$$

(fixed $D_i$)

Def. The multigraded semigroup of $X$.

$$\mathbb{Z}^{d+r}_{\geq 0} = \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}^r_{\geq 0}.$$

$$\Gamma(X) = \Gamma(X; D_1, \dots, D_r) = \{(\nu(s), \vec{m}) : 0 \neq s \in H^0(X, \mathcal{O}_X(\vec{m}D))\}.$$

$\Sigma(X) = \Sigma(\Gamma) \subseteq \mathbb{R}^{d+r}$ closed convex cone $(\Gamma(X))$

$$\Delta(X) = \Sigma(X) \subseteq \mathbb{R}^d \times \mathbb{R}^r.$$

$\Delta(X) \sim D_1, \dots, D_r \qquad N^1(X)_{\mathbb{R}} = \mathbb{R}^r$.

$\vec{a}$ s.t. $\vec{a}D$ is big $\quad \Sigma(X)$ over $\vec{a} \in \mathbb{R}^r$. $\quad \Delta(\vec{a}D)$.

$\Gamma \subseteq \mathbb{Z}^{d+r}_{\geq 0} = \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}^r_{\geq 0}$. $\qquad \vec{m} \cdot D \to \Delta(\vec{m}D) \subseteq \mathbb{R}^d$.

$\Sigma = \Sigma(\Gamma) \subseteq \mathbb{R}^{d+r}$. $\qquad N^1(X)_{\mathbb{R}} = \mathbb{R}^r$

$\Sigma \xrightarrow{pr_2} \mathrm{supp}(\Gamma) \subseteq \mathbb{R}^r$. $\qquad \mathbb{R}^d$. $\quad pr_1$. $\quad D$.

$\vec{a} \in \mathbb{Z}^r_{\geq 0}$, set.

$$\Gamma_{\mathbb{Z}_{\geq 0}\vec{a}} = \Gamma \cap (\mathbb{Z}^d_{\geq 0} \times \mathbb{Z}_{\geq 0}\vec{a}) \subseteq \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}_{\geq 0}\vec{a} = \mathbb{Z}^{d+1}_{\geq 0}$$

$$\Sigma_{\mathbb{R}\vec{a}} = \Sigma(\Gamma)_{\mathbb{R}\vec{a}} = \Sigma \cap (\mathbb{R}^d \times \mathbb{R}\vec{a})$$

$$\subseteq \mathbb{R}^d \times \mathbb{R}\vec{a}.$$

Prop. $\Gamma$ generates a subgroup of finite index in $\mathbb{Z}^{d+r}$.

$\vec{a} \in \mathbb{Z}^r_{\geq 0}$, $\vec{a} \in \mathrm{int}(\mathrm{supp}(\Gamma))$. Then

$$\Sigma(\Gamma_{\mathbb{Z}_{\geq 0}\vec{a}}) = \Sigma(\Gamma)_{\mathbb{R}\vec{a}}.$$

---

Prop A. $\Gamma$ generates subgroup of finite index in $\mathbb{Z}^n$.

$\to \Sigma$.

$L \subseteq \mathbb{R}^n$ defined over $\mathbb{Q}$. s.t. $L \cap \mathrm{int}(\Sigma)$

$L \subseteq \mathbb{R}^n$ defined over $\mathbb{Q}$. s.t. $L \cap \mathrm{int}(\Sigma)$

Then $\Sigma(T) \cap L = \Sigma(T \cap L)$.

Lemma. semigroup $\Gamma(X) \subseteq \mathbb{Z}_{\geq 0}^{d+r}$ gen $\mathbb{Z}^{d+r}$ as a group. $\square$

pf: $\mathrm{Big}(X) \underset{\text{open}}{\subset} N^1(X)_{\mathbb{R}}$, $\exists\, e_1, \dots, e_r \in N^1(X)_{\mathbb{R}}$. $\mathbb{Z}$-basis

$D_1, \dots, D_r$. $e_j$. $\mathbb{Z}_{\geq 0}$ - linear combination of $D$;

$$e_j \equiv \text{num}\ \underbrace{\vec{a_j} \cdot D}_{= E_j}. \qquad \vec{a_j} \in \mathbb{Z}^r$$

$\Gamma(E_j) \subset \Gamma(X)$. $a_1, \dots, a_r$ generates $\mathbb{Z}^r$ $\square$.

{ as group

$\mathbb{Z}^d \times \mathbb{Z} \cdot \vec{a_j}$

pf (of Thm):

$\Gamma = \Gamma(X; D_1, \dots, D_r)$

$\mathrm{supp}(\Gamma)$. spanned by $\vec{a} \in \mathbb{Z}^r = N^1(X)$. s.t.

$$H^0(X, \mathcal{O}_X(\vec{a}D)) \neq 0.$$

$\mathrm{Eff}(X)$. of $X$. $\cup$ interior $\mathrm{Big}(X)$ $\Big\} \Rightarrow \vec{a} \in \mathrm{int}(\mathrm{supp}(\Gamma))$ iff $\mathcal{O}_X(\vec{a}D)$ is big.

Given such $\vec{a}$.

$$\Gamma(X)_{\mathbb{N}\vec{a}} = \Gamma(\vec{a}D) \subseteq \mathbb{Z}_{\geq 0}^d \times \mathbb{Z}_{\geq 0} \vec{a}.$$

$\Delta(\vec{a}D)$ based on the cone $\Sigma(\Gamma_{\mathbb{Z}_{\geq 0}\vec{a}})$, i.e.

$\| \; \Sigma(\Gamma_{\mathbb{Z}_{\geq 0}\vec{a}}) \cap (\mathbb{R}^d \times \{\vec{a}\})$

$\Delta(X)_{\vec{a}}$ of $\Delta(X)$. over $\vec{a} \in \mathbb{R}^d$.

$\Delta(\xi) = \Delta(X)_\xi$ scale linearly with $\xi$. $\square$

4.3. Multi-Graded Linear Series.

$X$, irr. dim $d$. $D_1, \dots, D_r$ on $X$. $\vec{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$

$$\vec{m} D = \sum_{i=1}^{r} m_i D_i \quad , \quad |\vec{m}| = \sum_{i=1}^{r} |m_i|$$

$N^1_{\mathbb{R}}(S)$

$D \nearrow$

$\mathbb{Z}^r$

Def. —— $W_{\vec{\bullet}}$ on $X$ ~ $D_{\vec{\bullet}}$

$$W_{\vec{k}} \subseteq H^0(X, \mathcal{O}_X(\vec{k}D))$$

$\vec{k} \in \mathbb{Z}_{\geq 0}^r$, $W_{\vec{0}} = \mathbb{K}$, s.t.

$$\underbrace{W_{\vec{k}} \cdot W_{\vec{m}}}_{\wr} \subseteq W_{\vec{k}+\vec{m}}$$

$$(H^0(X, \mathcal{O}_X(\vec{k}D)) \otimes H^0(X, \mathcal{O}_X(\vec{m}D)) \longrightarrow H^0(X, \mathcal{O}_X((\vec{k}+\vec{m})D))$$

$(W_{\vec{k}} \otimes W_{\vec{m}}$.

$\vec{a} \in \mathbb{Z}_{\geq 0}^r$, $\underline{W_{\vec{a},\bullet}} \sim \vec{a} D$. $W_{k\vec{a}} \subseteq H^0(X, \mathcal{O}_X(k\vec{a}D))$.

$$\mathrm{vol}_{W_{\vec{\bullet}}}(\vec{a}) = \mathrm{vol}(W_{\vec{a},\bullet}).$$

$Y_\bullet$ on $X$. $\Delta(\vec{a}) = \Delta(W_{\vec{a},\bullet})$.

$$\mathrm{supp}(W_{\vec{\bullet}}) \subseteq \mathbb{R}^r.$$

spanned by $\vec{m} \in \mathbb{Z}_{\geq 0}^r$ s.t. $W_{\vec{m}} \neq 0$.

Def. $W_{\vec{\bullet}}$ satisfies (B') (or (C'))

(i). $\mathrm{supp}(W_{\vec{\bullet}}) \subseteq \mathbb{R}^r$. $\mathrm{int}(\mathrm{supp}(W_{\vec{\bullet}})) \neq \phi$,

(ii) $\forall \vec{a} \in \mathrm{int}(\mathrm{supp}(W_{\vec{\bullet}}))$,

$$W_{k\vec{a}} \neq 0 \quad k \gg 0.$$

(iii). $\exists \vec{a}_0 \in \mathrm{int}(\mathrm{supp}(W_{\vec{\bullet}}))$. s.t. $\mathbb{Z}_{\geq 0}$-graded $W_{\vec{a}_0,\bullet}$

Cond (B) (or (C)).

<u>Lemma</u> $W_{\vec{\bullet}}$ cond (B') or (C').

$\forall \vec{a} \in \mathrm{int}(\mathrm{supp}(W_{\vec{\bullet}}))$.

$\Rightarrow$ $W_{\vec{a},\bullet}$ satisfies Cond (B) or (C).

pf: [illegible]

$\Rightarrow$ $W_{\vec{a},\bullet}$ satisfies Cond (B) or (C).

pf: Cond (C'):

$m \gg 0$, $\exists F_{m\vec{a}_0}$ s.t.

$$m\vec{a}_0 D - F_{m\vec{a}_0} \equiv_{lin} A_{m\vec{a}_0}.$$

is ample;

$$\boxed{H^0(X, \mathcal{O}_X(pA_{m\vec{a}_0})) \subseteq W_{pm\vec{a}_0} \subseteq H^0(X, \mathcal{O}_X(pm\vec{a}_0 D))}$$

Loc $\vec{a} \in \mathrm{Int}(\mathrm{supp}(W_\bullet))$, $k \in \mathbb{Z}_{>0}$.

$$k\vec{a} = \vec{a}_0 + \vec{b} \in \mathrm{int}(\mathrm{supp}(W_\bullet)).$$

$W_{m\vec{b}} \neq 0$, $m \gg 0$:

$$E_{m\vec{b}} \sim \underset{\text{non-zero section}}{s_{m\vec{b}}} \in W_{m\vec{b}} \Rightarrow E_{m\vec{b}} \equiv_{lin} m\vec{b}D.$$

Then $mk\vec{a}D = m\vec{a}_0 D + m\vec{b}D$.

$$\underbrace{mk\vec{a}D - F_{m\vec{a}_0} - E_{m\vec{b}}}_{\text{ample}} \equiv_{lin} A_{m\vec{a}_0}.$$

$\forall p \gg 0$.

$$H^0(X, \mathcal{O}_X(pA_{m\vec{a}_0})) \subseteq W_{pm\vec{a}_0} \subseteq W_{pmk\vec{a}}$$

$s_{m\vec{b}}^{\otimes p}$.

$\overline{W}_{a,\bullet} \Rightarrow$ (i), (ii) Cond. (C).

(i) $\forall m \gg 0$. $\exists F_m \in \mathrm{eff}(X)$

$$A_m \equiv_{def} mD - F_m.$$

is ample

(ii) $\forall p \gg 0$.

$$H^0(X, \mathcal{O}_X(pA_m)) = H^0(X, \mathcal{O}_X(pmD - pF_m)) \subseteq W_{pm}$$
$$\subseteq H^0(X, \mathcal{O}_X(pmD)).$$

$\square$

Fix. $Y_\bullet$ on $X$.

$\in \mathbb{Z}_{>0}$,

Fix. $Y_\bullet$ on $X$.

Def. $W_{\vec{\bullet}}$ satisfies Cond(B'). w.r.t. $Y_\bullet$ if $\exists\ b \gg 0$ $\in \mathbb{Z}_{>0}$.

s.t. $\forall\ \vec{m} \in \mathbb{Z}^r_{\geq 0}$ and $\forall\ \underset{\neq 0}{s} \in W_{\vec{m}}$.

$$\nu_i(s) \leq b \cdot |\vec{m}| \quad \forall\ 1 \leq i \leq d.$$

□

multi-graded semigroup of $W_{\vec{\bullet}} \sim Y_\bullet$.

$$\Gamma(W_{\vec{\bullet}}) = \Gamma_{Y_\bullet}(W_{\vec{\bullet}}) = \{(\nu(s), \vec{m}) \mid 0 \neq s \in W_{\vec{m}}\} \subseteq \mathbb{Z}^{d+r}_{\geq 0}$$

Lemma. $W_{\vec{\bullet}}$. sat. Cond(B'). $\Rightarrow \exists\ Y_\bullet$ for $\Gamma_{Y_\bullet}(W_{\vec{\bullet}}) \xrightarrow[\text{as group}]{\text{gen}} \mathbb{Z}^{d+r}$

Cond(C') $\quad \forall\ Y_\bullet$. $\qquad \vec{a} \quad \Sigma(C)$.

pf: $\vec{a} \in \mathbb{Z}^r_{\geq 0} \longrightarrow$ supp$(W_{\vec{\bullet}})$.

$$\Gamma_{\vec{a}} = \Gamma_{Y_\bullet}(W_{\vec{a},\bullet}) \subseteq \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}_{\geq 0}\vec{a} \subseteq \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}^r_{\geq 0}$$

sub-semigroup $\to \Gamma(W_{\vec{\bullet}})$.

$\forall \vec{a}$. $\Gamma_{\vec{a}} \xrightarrow{\text{gen}} \mathbb{Z}^d \times \mathbb{Z}\vec{a}$

$$\begin{array}{ccccc} \vec{a}_1, & \cdots & \vec{a}_r & \to & \mathbb{Z}^r \\ \downarrow & & \downarrow & & \\ \Gamma_{\vec{a}_1} & \cdots & \Gamma_{\vec{a}_r} & \longrightarrow & \mathbb{Z}^{d+r} \end{array}$$

□

$$\Sigma(W_{\vec{\bullet}}) \subseteq \mathbb{R}^d \times \mathbb{R}^r.$$

span

$\Gamma(W_{\vec{\bullet}})$.

$$\Delta(W_{\vec{\bullet}}) = \Sigma(W_{\vec{\bullet}}).$$

$$\Delta(W_{\vec{\bullet}}) \subseteq \mathbb{R}^d \times \mathbb{R}^r$$

$$\searrow \ \mathbb{R}^r \ \swarrow pr_2.$$

Thm. $W_{\vec{\bullet}}$ sat (A'), (B'). (C'), $Y_\bullet$

$\forall\ \vec{a} \in \operatorname{int}(\operatorname{supp}(W_{\vec{\bullet}}))$.

fibre of $\Delta(W_{\vec{\bullet}})$ over $\vec{a}$. $\sim$ $W_{\vec{a},\bullet}$

$$\Delta(W_{\vec{\bullet}})_{\vec{a}} = \Delta(\vec{a})$$

$$\Delta(W_{\vec{z}})_{\vec{a}} = \Delta(\vec{a})$$