1.2. Construction Δ.

D X ← projective $H^0(X, \mathcal{O}_X(mD))$ ← finite dim.

$\nu_{Y_\bullet}$ $Y_\bullet$ $\mathcal{O}_X(D) \xrightarrow[\nu_{Y_\bullet}]{\text{gen}}$ add semigroup $\subseteq \mathbb{N}^d$.

$\ni x \longrightarrow \underline{\nu_{Y_\bullet}(x)}$.

Def (graded)

$$\Gamma(D) = \Gamma_{Y_\bullet}(D) = \{ (\underbrace{\nu_{Y_\bullet}(s)}_{d\text{-tuple}}, m) \mid 0 \neq s \in H^0(X, mD)), m \geq 0 \}$$

$$\subseteq \mathbb{N}^d \times \mathbb{N} = \mathbb{N}^{d+1}$$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$.

$\Gamma(D) \subseteq \mathbb{Z}^{d+1} \subseteq \mathbb{R}^{d+1}$ via standard inclusions. → volume

- closure → $\underline{\mathrm{vol}(\Delta)} \subseteq \mathbb{R}^d$.
- limit point

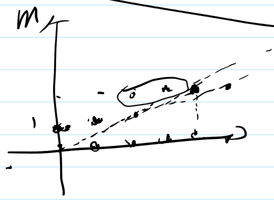
eg (Failure of f.g.). $m \to \infty$

D, C curve, g. $\deg(D) = \underline{c \geq 2g+1}$. $C \supseteq \{p\}$.

$\Gamma(D) = \{(0,0)\} \cup \{ (k, m) \mid m \geq 1, \underline{0 \leq k \leq mc - g} \}$ ↗ ∞

Riemann-Roch

$g = 1$.



m. □

$\Gamma = \Gamma(D)$. $\xrightarrow{\text{closed convex cone, vertex at } 0}$ $\underline{\Sigma(\Gamma)} \subseteq \mathbb{R}^{d+1}$.

Def. (Newton-Okounkov body). D, $Y_\bullet$

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \Sigma(\Gamma) \cap (\mathbb{R}^d \times \{1\}) \subseteq \mathbb{R}^d.$$

compact closed, convex subset

$$\Gamma(D)_m = \mathrm{Im}\left((H^0(X, \mathcal{O}_X(mD)) - \{0\}) \xrightarrow{\nu} \mathbb{Z}^d \right).$$

Then.

$$\underline{\Delta(D)} = \text{closed convex hull}\left(\underline{\bigcup_{m \geq 1} \tfrac{1}{m} \cdot \Gamma(D)_m} \right) \subseteq \mathbb{R}^d.$$

non-negative orthant of $\mathbb{R}^d$.

Rmk (Line bundles). $\mathcal{O}_X(D) = L$.

Rmk: (Line bundles). $\mathcal{O}_X(D) = L$.

Lemma. (Boundness) $\Delta(D) \subseteq$ bounded subset of $\mathbb{R}^d$.

Pf: $b \gg 0$
$v_i(s) < mb$, $1 \le i \le d$, $m > 0$, $0 \ne s \in H^0(X, \mathcal{O}_X(mD))$

Fix an ample H, b_1
$$(D - b_1 Y_1) \cdot H^{d-1} < 0.$$

$\xi \in N^1(X)$

$\Rightarrow \forall s$, $v_1(s) \le mb_1$

$D_1 \equiv_{num} D_2$
$\Downarrow$
$\Delta(D_1) = \Delta(D_2)$

Next, b_2 on Y_1
$$((D - aY_1)|_{Y_1} - b_2 Y_2) \cdot H^{d-2} < 0.$$

Ampleness num. inv?

$\forall\ 0 \le a \le b_1$.

$s \in H^0(X, \mathcal{O}_X(mD))$. $v_2(s) \le mb_2$.

$\{b_i\}$. $v_i(s) < mb_i$ $\quad b = \max\{b_i\}$ $\quad \square$

Rmk. (Several divisors)
$D_1, \cdots, D_r$ on X. assert $\exists\ b \gg 0$ with $v_i(s) \le b \cdot \sum |m_j|$.
$m_1, \cdots, m_r$. $\forall\ 0 \ne s \in H^0(X, \mathcal{O}_X(m_1 D_1 + m_2 D_2 + \cdots + m_r D_r))$.

In fact. $b_1 > 0$.
$$(\textstyle\sum \lambda_i D_i - b_1 Y_1) \cdot H^{d-1} < 0.$$

where $\sum|\lambda_i| \le 1 \Rightarrow v_1(s) < b_1 \cdot \sum |m_j|$.

$b_2 > 0$.
$$((\textstyle\sum \lambda_i D_i - aY_1)|_{Y_1} - b_2 Y_2) \cdot H^{d-2} < 0.$$

$v_2(s) < b_2 \cdot \sum |m_j|$. $\square$

Rmk: $\forall D$. $\exists$ $\Delta(D) \subseteq \mathbb{R}^d$ $\quad \mathrm{int}(\Delta(D)) = \emptyset$.
$\times$ convex body.
$D \ge 0$.

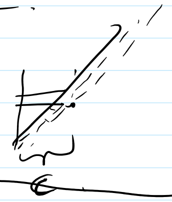
D big. pseudo eff

Eg (Curves). D. $\deg = c > 0$. on smooth C, g.

$$\underline{\Delta(D) = [0, c] \subset \mathbb{R}}.$$

$\underline{mc - g}$.

$m \to \infty$.



Eg $X = \mathbb{P}^d$, $\underline{D = H}$. $\boxed{Y_i = \{ \underline{X_1 = X_2 = \cdots = X_i} = 0 \}}$.

$\underline{\Delta(D)}$ simplex

$= \{ (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d ; \xi_1 \geq 0 \ldots \xi_d \geq 0, \sum \xi_i \leq 1 \}$.

$\forall m$, $s \in \underline{H^0(X, \mathcal{O}_X(mD))}$. $\underline{\Gamma(D)_m} = \{ \xi_i \geq 0, \sum \xi_i = m \} \subseteq \mathbb{N}^d$.

$d = 2$. $m = 1$. $\underline{m \to \infty}$. $\underline{\text{simplex } \mathbb{R}^d}$.

$\underline{X_0^{a_0} X_1^{a_1} X_2^{a_2}}$. (monomial)

moment polytope.

☆ N.O. div. of Toric with $Y_\cdot$ moment polytope of $\underline{\text{Div}}$.

Lazar. Mustaţă.

1.2. Graded Linear Series

D. on X. (X complete) $W_\cdot = \{ W_k \} \sim D$.

$W_k \subseteq H^0(X, \mathcal{O}_X(kD))$ finite dimension $\forall k$.

$W_0 = \mathbb{K}$.

$(*)$ $\underbrace{W_k \cdot W_l}_{\mathrm{H}} \subseteq W_{k+l}$

$(H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(lD)) \longrightarrow H^0(X, \mathcal{O}_X((k+l)D))$

$(W_k \otimes W_l)$

$R(W_\cdot) = \oplus W_n$ graded subalg $\subseteq R(X, D) = \oplus H^0(X, \mathcal{O}_X(mD))$.

Def $W_\cdot$ on X, D. graded semigroup of $W_\cdot$.

$$\Gamma(W_\cdot) = \Gamma_{Y_\cdot}(W_\cdot) = \{ (\nu_{Y_\cdot}(s), m) \mid 0 \neq s \in \underline{W_m}, m \geq 0 \} \subseteq \mathbb{Z}^{d+1}$$

The N.O. body of $W_\cdot$ $\Gamma = \Gamma(W_\cdot)$

$\ldots \cap (\mathbb{R}^d \times \{1\})$

The N.O. body of $W_\bullet$ $\quad \Gamma = \Gamma(W_\bullet)$

$\Delta(W_\bullet) = \Delta_{Y_\bullet}(W_\bullet) = $ closed convex cone $(\Gamma) \cap (\mathbb{R}^d \times \{1\})$ $\square$

$\Gamma(W_\bullet)$. closed convex subset of $\mathbb{R}^d$.

$\Delta(W_\bullet) = $ closed convex hull $\left(\bigcup_{m\geq 1} \frac{1}{m} \cdot \Gamma(\underline{W_\bullet})_m \right) \subseteq \mathbb{R}^d$.

Rmk $\quad X = \mathbb{A}^1,\ D = 0.\ \ \underline{W_m} = \{ f \mid \deg(f) \leq m^2 \}$.

Argument of $\underline{\text{vol}}$. constant w.r.t. m. $\quad$ Bound.

$\rightsquigarrow m^2$.

$\Delta \xleftarrow{\text{con}}$ var $\quad$ (compact)

Prop. $\underline{K} \subseteq \mathbb{R}^d$. convex body. possibly translation and scaling $\underline{\mathbb{K}}$. $\exists\ W_\bullet$ on $\mathbb{P}^d \sim H,\ Y_\bullet$ s.t.

$$K = \Delta(W_\bullet)$$

pf: $\quad T \subseteq \mathbb{R}^d$. $\underline{\text{simplex}}$. $\quad$ exponent vector $\Rightarrow x^\nu$. in var $\{x_1, \cdots, x_d\}$

assume $K \subseteq T$. $\quad \underset{\sim}{\nu} \in mT \cap \mathbb{Z}^d \quad \Rightarrow x^\nu \quad \deg(x^\nu) \leq m$.

$$W_m' = \operatorname{span}_{\mathbb{K}} \langle x^\nu : \nu \in mK \cap \mathbb{Z}^d \rangle.$$

Then $\quad W_k' \cdot W_\ell' \subseteq W_{k+\ell}'$. for $m, \ell \geq 0$.

W_m' determines by $\quad W_m \subseteq H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(m))$.

$\Rightarrow W_\bullet$

If $\underline{Y_\bullet}$. $\quad \Gamma(W_\bullet)_m = mK \cap \mathbb{Z}^d$.

$\Gamma_m = \operatorname{Im}(\nu(W_m - \{0\}) \longrightarrow \mathbb{Z}^d)$.

$$\Rightarrow \bigcup_{m\geq 1} \frac{1}{m} \cdot \Gamma(W_\bullet)_m = K \cap \mathbb{Q}^d$$

$\underline{K = \text{closure}(K \cap \mathbb{Q}^d)} = \Delta(W_\bullet)$. $\quad \square$

2. Volumes of N.O. body.

2.1. $\Gamma \subseteq \mathbb{N}^{d+1}$

2.1. $\Gamma \subseteq \mathbb{N}^{d+1}$, ~~assume f.g~~

$$\Sigma = \Sigma(\Gamma) = \text{closed convex cone}(\Gamma)$$

$$\Delta = \Delta(\Gamma) = \Sigma \cap (\mathbb{R}^d \times \{1\}).$$

$\forall m \in \mathbb{N}$.

$$\Gamma_m = \Gamma \cap (\mathbb{N}^d \times \{m\}) \subseteq \mathbb{N}^d.$$

(i). $\Gamma_0 = \{0\} \in \mathbb{N}^d$.

(ii). $\exists$ finite $(v_i, 1)$ span semigroup $B \subseteq \mathbb{N}^{d+1}$ s.t.

$$\Gamma \subseteq B.$$

(iii). $\Gamma \xrightarrow[\text{as group}]{\text{gen}} \mathbb{Z}^{d+1}$.

Prop: Γ sat (i)~(iii) Then.

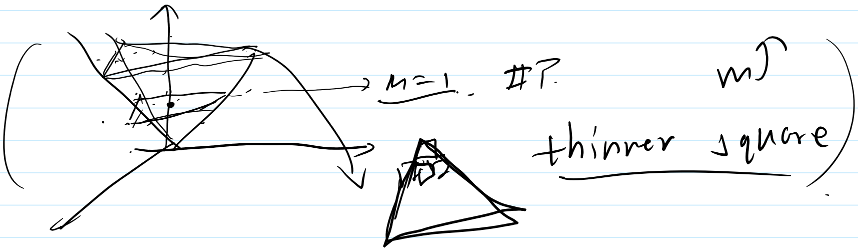
$$\lim_{m\to\infty} \frac{\#\Gamma_m}{m^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta)$$

(eg. $\mathrm{vol}_{\mathbb{R}^d}([0,1]^d) = 1$)

pf: $\Gamma_m \subseteq m\Delta \cap \mathbb{Z}^d$. & $\boxed{\lim_{m\to\infty} \frac{\#(m\Delta \cap \mathbb{Z}^d)}{m^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta)}$

$\Rightarrow$. $\limsup_{m\to\infty} \frac{\#\Gamma_m}{m^d} \leq \mathrm{vol}_{\mathbb{R}^n}(\Delta)$.

$\#\Gamma_m \leq \#(m\Delta \cap \mathbb{Z}^d)$



Reverse.

1. (Γ f.g), by Khovanskii, $\exists\, y \in \Gamma$ s.t.

$$(\Sigma + y) \cap \mathbb{N}^{d+1} \subseteq \Gamma.$$

$\because$ Γ generates $\mathbb{Z}^{d+1}$ as group by (iii).

$$\lim_{m\to\infty} \frac{\#\{(\Sigma+y) \cap (\mathbb{N}^d \times \{m\})\}}{m^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta)$$

$\Rightarrow$. $\liminf_{m\to\infty} \frac{\#\Gamma_m}{m^d} \geq \mathrm{vol}_{\mathbb{R}^d}(\Delta)$

$m \to \infty \quad m^{d}$

2. $\underbrace{\Gamma^1 \subseteq \Gamma^2 \subseteq \cdots}_{\text{f.g.}} \subseteq \Gamma$. $\quad \Gamma^i$ sat (i) ~ (iii)

$\Gamma = \bigcup_i \Gamma^i$. $\quad \#\Gamma_m \geq \#(\Gamma^i)_m, \ \forall m \in \mathbb{N}$.

$\Delta^i = \Delta(\Gamma^i)$.

$$\liminf_{m \to \infty} \frac{\#\Gamma_m}{m^d} \geq \mathrm{vol}_{\mathbb{R}^d}(\Delta^i) = \lim \cdots \quad \forall i.$$

But $\mathrm{vol}_{\mathbb{R}^d}(\Delta^i) \longrightarrow \mathrm{vol}_{\mathbb{R}^d}(\Delta)$. holds for Γ itself $\square$.

2.2. Global Linear Series.

Lemma: Let X prj $\dim = d$, $\underline{Y_\bullet}$, $D \in \mathrm{big}\,\mathrm{Div}(X)$

$$\Gamma = \Gamma_{Y_\bullet}(D) \subseteq \mathbb{N}^{d+1}$$

satisfies (i), (ii), (iii)

pf: (i) $\Gamma_0 = \{0\}$.

(continue).

vol (Big div)
Sec 3 wrong.
Chap 3 $\underline{\text{Fujita}}$. x.
Chap 4. div
5. flag
chap 6. Toric eg.

Next time: Mar. 19. 2024. $\qquad$ O.K. Succ in Surface.

- Kuron, Lozovanu,Maclean 2010: convex bodies appearing as okounkov bodies of divisors
- Patrycja Luszcz-Swidecka and David Schmitz 2013: Minkowski decomposition of Okounkov bodies on surfaces