X irreducible variety of dim d.

Admissible flag: local equation

$$Y_\bullet : X = \underbrace{Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_d}_{} = \{pt\}$$

irreducible subvarieties of X, $\operatorname{codim}(Y_i) = i$.
each Y_i is nonsingular at Y_d.

$Y_\bullet$, $\Delta \subseteq \mathbb{R}^d \sim D$ on X (complete)

① $Y_\bullet$, valuation-like function
② Δ is built from ...

1.1 Valuation

$\forall D$ on X

$$v = v_{Y_\bullet} = v_{Y_\bullet, D} : H^0(X, \mathcal{O}_X(D)) \longrightarrow \mathbb{Z}^d \cup \{\infty\}$$
$$s \longmapsto v(s) = (v_1, \ldots, v_d)$$

s.t.

① $v_{Y_\bullet}(s) = \infty$ iff $s = \infty$

② $\mathbb{Z}^d$ - lexicographically order, &

$$v_{Y_\bullet}(s_1 + s_2) \geq \min\{v_{Y_\bullet}(s_1), v_{Y_\bullet}(s_2)\}$$

③ $s \in H^0(X, \mathcal{O}_X(D))$, $t \in H^0(X, \mathcal{O}_X(E))$.

$$v_{Y_\bullet, D+E}(s \otimes t) = v_{Y_\bullet, D}(s) + v_{Y_\bullet, E}(t).$$

$Y_{i+1} \in \operatorname{CDiv}(Y_i)$, Y_i smooth at Y_d.

Given

$0 \neq s \in H^0(X, \mathcal{O}_X(D))$.

$v_1 = v_1(s) = \operatorname{ord}_{Y_1}(s)$

$\Rightarrow \bar{s}_1 \in H^0(X, \mathcal{O}_X(D - v_1 Y_1))$

$\Rightarrow s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - v_1 Y_1))$

$\Rightarrow v_2 = v_2(s) = \operatorname{ord}_{Y_2}(s_1)$

$a_1, \ldots, a_i \geq 0$ $\mathcal{O}(D - a_1 Y_1 - \cdots - a_i Y_i)|_{Y_i}$

$k[x_1, \ldots, x_n]$

$F_i \sim Y_i$

$s \leftrightarrow f = F_1^{v_1} G_1$

$= F_1^{v_1}(F_1^{u_1} R_1 + F_2^{v_2} G_2)$

$= F_1^{v_1}(F_1^{u_1} R_1 + F_2^{v_2}(F_2^{u_2} R_2 + F_3^{v_3} G_3))$

$\vdots$

$f_1 = f // F_1^{v_1} = F_1^{u_1} R_1 + F_2^{v_2} G_2.$

$a_1, \cdots, a_i \geq 0$, $\mathcal{O}(D - a_1Y_1 - \cdots - a_iY_i)|_{Y_i}$ the line bundle

$\mathcal{O}_X(D)|_{Y_i} \otimes \mathcal{O}_X(-a_1Y_1)|_{Y_i} \otimes \cdots \otimes \mathcal{O}_{Y_{i-1}}(-a_iY_i)|_{Y_i}$

on Y_i

$f_1 = f // \overline{F_1}^{\nu_1} = \overline{F_1}^{u_1} R_1 + \overline{F_2}^{\nu_2} G_2.$

$\overline{f_1} = f_1 (\mathrm{mod}\, F_1) = \overline{F_2}^{\nu_2} \overline{G_2}$

$f_2 = \overline{f_1} // \overline{F_2}^{\nu_2} = \overline{G_2} = \overline{F_2}^{u_2} \overline{R_2} + \overline{F_3}^{\nu_3} \overline{G_3}$

$\overline{f_3} = f_2 (\mathrm{mod}\, \overline{F_2}) = \overline{\overline{F_3}}^{\nu_3} \overline{\overline{G_3}}$

$i \leq k \leq d$ construct non-vanishing section.

step

$s_i \in H^0(Y_i, \mathcal{O}(D - \nu_1Y_1 - \cdots - \nu_iY_i)|_{Y_i})$

with $\nu_{i+1}(s) = \mathrm{ord}_{Y_{i+1}}(s_i)$, so that

$$\nu_{k+1}(s) = \mathrm{ord}_{Y_{k+1}}(s_k)$$

$$\Rightarrow \widetilde{s_{k+1}} \in H^0(Y_k, \mathcal{O}(D - \nu_1Y_1 - \nu_2Y_2 - \cdots - \nu_kY_k)|_{Y_k} \otimes \mathcal{O}_{Y_k}(-\nu_{k+1}Y_{k+1})$$

$$s_{k+1} \quad Y_{k+1} \quad (D - \nu_1Y_1 - \cdots - \nu_{k+1}Y_{k+1})|_{Y_{k+1}}$$

$s_i \sim$ local equation of Y_i in Y_{i-1}

Eg: $X = \mathbb{P}^d$, $Y_i = V(T_1 = \cdots = T_i = 0)$

$\mathcal{O}_X(D) \cong \mathcal{O}_X(m)$

$$T_0^{a_0} T_1^{a_1} \cdots T_d^{a_d} \xrightarrow{\nu_{Y_\cdot}} (a_1, \cdots, a_d)$$

Eg: C, g, $p \in C$, $C \supseteq \{p\}$. D on C

$$\nu: H^0(C, \mathcal{O}_X(D)) - \{0\} \longrightarrow \mathbb{Z}$$

$c = \deg(D) \geq 2g+1$, $[0, c] \Rightarrow \mathrm{Im}(\nu) = \{0, 1, \cdots, c-g\}$

Lemma. $W \subseteq H^0(X, \mathcal{O}_X(D))$. Fix $a = (a_1, \cdots, a_d) \in \mathbb{Z}^d$.

Set: $W_{\geq a} = \{s \in W : \nu_{Y_\cdot}(s) \geq a\}$. $W_{>a} = \{s \in W ; \nu_{Y_\cdot}(s) > a\}$

Then $\dim(W_{\geq a}/W_{>a}) \leq 1$.

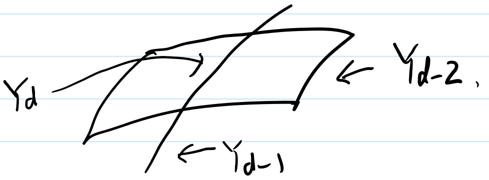
W finite dim.

$$\#(\mathrm{im}(W - \{0\}) \xrightarrow{\nu} \mathbb{Z}^d) = \dim W.$$

$\# (\operatorname{im} (W - \{0\}) \xrightarrow{v} \mathbb{Z}^d) = \dim W.$

pf1:

$$\mathcal{O}(D - a_1 Y_1 - a_2 Y_2 - \cdots - a_{d-1} Y_{d-1})|_{Y_{d-1}} \otimes \frac{\mathcal{O}_{Y_{d-1}}(-a_d Y_d)}{\mathcal{O}_{Y_{d-1}}(-(a_d+1) Y_d)}$$



$\square$

pf2: Alg

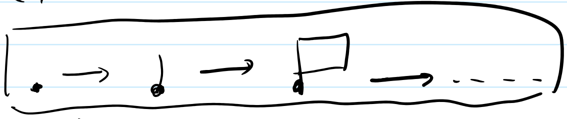
	$v(f)=v(g)$	$v(f)\neq v(g)$
	$v(f+g) > \min(v(f), v(g))$	$v(f+g) = \min\{v(f), v(g)\}$

X, $W_{\geq a}$, $W_{>a}$. $m = x_1^{a_1} \cdots x_d^{a_d}$

$v = a$ { ① $s \in W$. monomial $\times c$ uniquely.
② $s = m_1 + m_2 \Rightarrow v(m_1) = a$.

$\Rightarrow W_{\geq a}/W_{>a} = \langle x_1^{a_1} \cdots x_d^{a_d} \rangle$ or $\langle \emptyset \rangle$.

$\square$

Rmk: (Partial flag)  Why call it "flag"

$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_d = \{pt\}$

$Y'_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots Y_r$. $\operatorname{codim}(Y_i) = i$

$\hookrightarrow Y_i$ non-singular at generic point of Y_r.

D, $v_{Y'_\bullet} : H^0(X, \mathcal{O}_X(D)) \longrightarrow \mathbb{Z}^r \cup \{\infty\} \subseteq \mathbb{Z}^d \cup \{\infty\}$

s.t. (i) $\sim$ (iii)

Rmk (Sheafification)

L linebundle on X, D, $v_{Y_\bullet}$.

fix $\sigma = (\sigma_1, \cdots, \sigma_d) \in \mathbb{Z}^d$, $\exists$ coherent sheaf $L^{\geq \sigma} \subseteq L$. by

$$L^{\geq \sigma}(U) = \{ s \in L(U) ; v_{Y_\bullet}|_U(s) \geq \sigma \},$$

$\forall$ open set $U \subseteq X$.

$Y_{i+1} \in \operatorname{CDiv}(Y_i)$ $\qquad$ $L^{\geq \sigma}$ can be constructed iteratively.

$L^{\geq(\sigma_1)} = L(-\sigma_1 Y_1)$

$$L^{\geq(\sigma_1)} = L(-\sigma_1 Y_1)$$

$\Rightarrow$ $L^{\geq(\sigma_1,\sigma_2)}$ inverse image of $L(-\sigma_1 Y_1 - \sigma_2 Y_2)|_{Y_1} \subseteq L(-\sigma_1 Y_1)|_{Y_1}$

under the surjection $L(-\sigma_1 Y_1) \to L(-\sigma_1 Y_1)|_{Y_1}$

$$\begin{tikzcd}
L^{\geq(\sigma_1,\sigma_2)} \arrow[r, two heads] \arrow[d, hook] & L(-\sigma_1 Y_1 - \sigma_2 Y_2)|_{Y_1} \arrow[d, hook] \\
L^{\geq(\sigma_1)} = L(-\sigma_1 Y_1) \arrow[r, two heads] & L(-\sigma_1 Y_1)|_{Y_1}
\end{tikzcd}$$

$L^{\geq(\sigma_1 \cdots \sigma_d)}$, each $Y_{i+1} \subseteq Y_i$

Open neighbourhood $j: V \subseteq X$ of Y_d and put.

$$L^{\geq\sigma} = j_*(L|_V)^{\geq\sigma}) \cap L.$$

$L \otimes \underbrace{K(X)}_{\text{field.}} \to$ defined by the stalk of L at generic pt of X.

Next Time : Mar. 5, 2024.